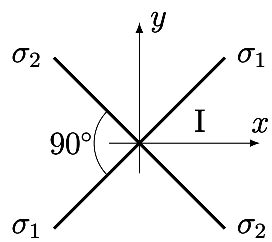
Physics 5C - Final

Monday, June 9, 3-6PM

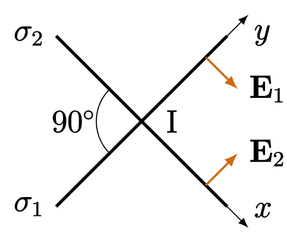
UCLA / Spring 2025 / Brian Naranjo

Solutions

1) (10 points) Two uniformly-charged infinite planes, carrying positive surface charge densities $\sigma_1 = 3\sigma_0$ and $\sigma_2 = 4\sigma_0$, intersect along the z-axis, separating space into four regions. In terms of σ_0, find the electric field's *magnitude* in region I.



Solution We are always free to choose to work in whatever coordinate system is most convenient. In this problem, it is natural to choose axes parallel to the planes,



Then, the field components are

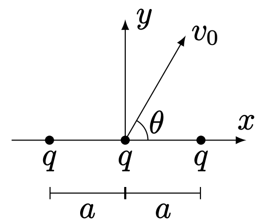
$$\begin{aligned} E_{1x} &= \frac{\sigma_1}{2\epsilon_0} & E_{2x} &= 0 \\ E_{1y} &= 0 & E_{2y} &= \frac{\sigma_2}{2\epsilon_0} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} E_x &= E_{1x} + E_{2x} = \frac{3\sigma_0}{2\epsilon_0} \\ E_y &= E_{1y} + E_{2y} = \frac{4\sigma_0}{2\epsilon_0} \end{aligned}$$

and the field magnitude is

$$E = \sqrt{E_x^2 + E_y^2} = \frac{\sigma_0}{2\epsilon_0}\sqrt{3^2 + 4^2} = \boxed{\frac{5\sigma_0}{2\epsilon_0}}$$

■

2) (10 points) Three identical particles of mass m and charge $q > 0$ are located on the x-axis. The particles located at $x = \pm a$ remain stationary. The particle located at the origin initially has velocity v_0 directed at an angle θ above the x-axis. Find its speed v_1 in the limit as it moves infinitely far from the origin.



Solution Throughout the particle's motion, total mechanical energy is conserved,

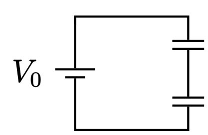
$$U_0 + K_0 = U_1 + K_1 \implies \cancel{\frac{kq^2}{2a}} + 2\frac{kq^2}{a} + \frac{1}{2}mv_0^2 = \cancel{\frac{kq^2}{2a}} + \frac{1}{2}mv_1^2.$$

Note that the potential energy due to the interaction between the two fixed charges, $kq^2/2a$, remains constant and could have been neglected in our calculation. Solving for v_1,

$$\boxed{v_1 = \sqrt{v_0^2 + \frac{4kq^2}{ma}} = \sqrt{v_0^2 + \frac{q^2}{\pi\epsilon_0 ma}}}$$

You can think of this as the initial potential energy of the interaction between the mobile charge and the two fixed charges, $2kq^2/a$, being totally converted to *additional* kinetic energy as the mobile charge flies off to infinity. ■

3) (10 points) Two identical parallel-plate capacitors, each having gap d_0, are connected to battery V_0, as shown. The total energy stored in the capacitors is U_0. We then entirely fill *one* of the capacitor's gaps with dielectric κ while increasing the *other* capacitor's gap to d_1. Find d_1 such that the total energy stored doesn't change, $U_1 = U_0$.



Solution Take the initial individual capacitance values to be C. The initial equivalent capacitance of the two capacitors wired in series is

$$C_0 = \frac{C \cdot C}{C + C} = \frac{C}{2}.$$

Using $C_{\text{plate}} = \kappa\epsilon_0 A/d$, the individual capacitors change as $C \to \kappa C$ and $C \to d_0 C/d_1$. Then, the final equivalent series capacitance is

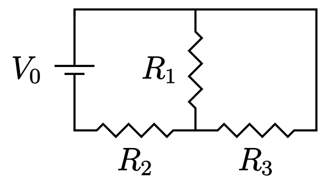
$$C_1 = \frac{(\kappa C)\cdot(d_0 C/d_1)}{\kappa C + (d_0 C/d_1)} = C\frac{\kappa \cdot (d_0/d_1)}{\kappa + (d_0/d_1)} = C\frac{\kappa d_0}{\kappa d_1 + d_0}.$$

Using $U = CV_0^2/2$,

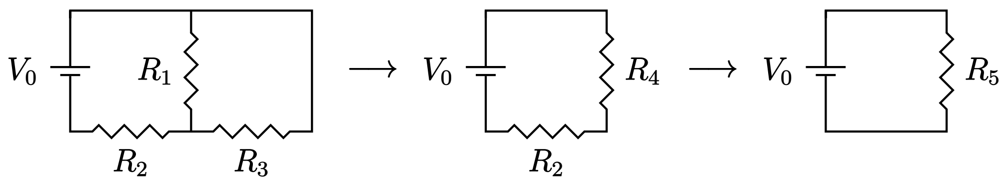
$$1 = \frac{U_1}{U_0} = \frac{C_1}{C_0} = \frac{2\kappa d_0}{\kappa d_1 + d_0} \implies \kappa d_1 + d_0 = 2\kappa d_0 \implies \boxed{d_1 = d_0\left(2 - \frac{1}{\kappa}\right) = d_0\left(\frac{2\kappa - 1}{\kappa}\right)}$$

As a consistency check, $\kappa = 1$ gives $d_1 = d_0$, as expected. ■

4) (10 points) Take $V_0 = 12$ V, $R_1 = 3\ \Omega$, $R_2 = 2\ \Omega$, $R_3 = 6\ \Omega$. Find each resistor's potential difference, current, and power dissipation.



Solution We use the tabular method. Resistors R_1 and R_3 are in parallel, so we replace them with resistor R_4. Then, resistors R_2 and R_4 are in series, so we replace them with resistor R_5.



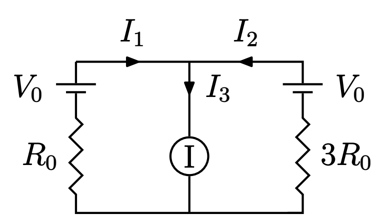
i	V_i (V)	I_i (A)	R_i (Ω)	P_i (W)
1	6	2	3	12
2	6	3	2	18
3	6	1	6	6
4	6	3	2	18
5	12	3	4	36

Steps I took in filling out the table:

1. R_1, R_2, and R_3 are given.
2. $R_4 = R_1R_3/(R_1 + R_3)$
3. $R_5 = R_2 + R_4$
4. $V_5 = V_0 = 12$ V
5. $V_5 = I_5R_5$. In a row, whenever we see that two out of three V, I, or R columns are determined, we can find the third value via $V = IR$.
6. $I_2 = I_4 = I_5$. Series resistors carry the same current as their equivalent resistor.
7. Determine V_2 and V_4 via $V = IR$.
8. $V_1 = V_3 = V_4$. Parallel resistors have the same potential difference as their equivalent resistor.
9. Determine I_1 and I_3 via $V = IR$.
10. $P_i = V_iI_i$

To check our work, we confirm that $P_1 + P_2 + P_3 = P_2 + P_4 = P_5$. ■

5) (10 points) In the circuit below, *current source* I maintains a steady current of I_3 in the middle branch. Find I_1 and I_2 in terms of I_3.



Solution We have two unknowns, I_1 and I_2, so we need two independent equations.

1. The junction rule gives

$$\sum I_{\text{in}} = \sum I_{\text{out}} \implies I_1 + I_2 = I_3$$

2. For a clockwise loop around the circuit's perimeter, the loop rule gives

$$\sum(\Delta V)_i = 0 \implies -I_1 R_0 + V_0 - V_0 + 3I_2 R_0 = 0 \implies I_1 = 3I_2$$

Solving these two equations yields $\boxed{I_1 = 3I_3/4}$ and $\boxed{I_2 = I_3/4}$.

Why did we not choose the left or right loops? Recall that *voltage* sources provide a varying current while maintaining a constant potential difference, and *current* sources provide a varying potential difference while maintaining a steady current. We went around the circuit's perimeter to avoid the unknown potential difference across the current source.

If we insist upon using the the loop rule around the circuit's left loop, then we must introduce the potential difference V_I across the current source. Taking V_I to be polarized in the direction of I_3, traversing clockwise,

$$-I_1 R_0 + V_0 + V_I = 0.$$

Now that we have an extra unknown, we need an additional independent equation. Traversing the right loop counterclockwise,

$$-3I_2 R_0 + V_0 + V_I = 0.$$

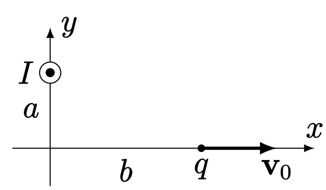
Subtracting these two equations,

$$-I_1 R_0 + 3I_2 R_0 = 0,$$

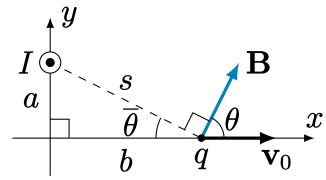
which is the same as the equation obtained from going around the circuit's perimeter, finally leading to the solution obtained more directly above.

■

6) (10 points) Infinite straight-line current I is parallel to the z-axis and intercepts the positive y-axis at $y = a$. A particle of charge $q > 0$, located on the positive x-axis at $x = b$, has velocity v_0 along the positive x-direction. The magnetic force $\mathbf{F}$ acting on q is directed out-of-the-page. What is the magnitude of this force?



Solution Using the right-hand rule, the magnetic field circulates counter-clockwise about I,

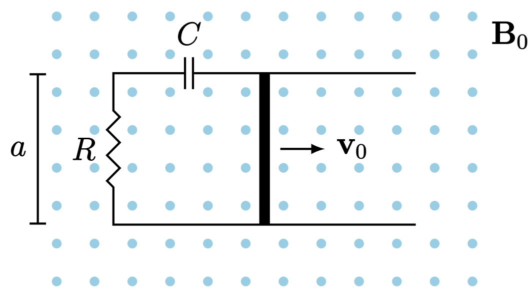


Angles θ and $\overline{\theta}$ are *complementary*, which allows us to easily evaluate $\sin\theta$ in terms of the right-triangle,

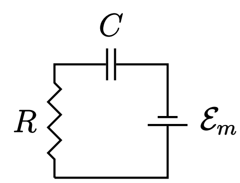
$$F = qv_0 B \sin\theta = qv_0 B \cos\overline{\theta} = qv_0 \frac{\mu_0 I}{2\pi s} \cdot \frac{b}{s} = \boxed{\frac{qv_0 \mu_0 I b}{2\pi(a^2 + b^2)}}$$

where we used the Pythagorean theorem, $s^2 = a^2 + b^2$. ■

7) (10 points) A slidewire circuit, of resistance R and capacitance C, is immersed in a uniform static magnetic field $\mathbf{B}_0$, which is directed out-of-the-page. The slidewire, of length a, is pulled to the right at constant speed v_0. Initially, at time $t = 0$, the capacitor is uncharged. As a function of time, find the force $F(t)$ with which you must pull on the slidewire to maintain its speed.



Solution The magnetic field does not vary over time, so the induced emf is purely motional. We could find the magnitude of the emf using Faraday's flux rule, but we recognize our standard slidewire configuration, where the slidewire's velocity $\mathbf{v}_0$, the slidewire's direction along its length a, and the magnetic field $\mathbf{B}_0$ are all mutually perpendicular. From the cheat sheet, the resulting motional emf is $\mathcal{E}_m = v_0 a B_0$. Furthermore, Lenz's law indicates that the induced emf is polarized *clockwise*, so that the induced current's magnetic field points into the plane. The equivalent circuit is



This is our basic RC charging circuit. Initially, when the capacitor is uncharged, the potential difference across resistor R is equal to $\mathcal{E}_m$. Using Ohm's law, the initial current is $I_0 = \mathcal{E}_m/R$. As the capacitor charges, the current through the circuit decays with time constant $\tau = RC$,

$$I(t) = I_0 e^{-t/\tau}.$$

Because the current flows clockwise, magnetic force acting on the slidewire is to the left. To maintain the slidewire's constant speed, this magnetic force must be balanced by the force of our hand pulling the slidewire toward the right. The magnitude of this force is thus

$$F(t) = I(t) a B_0 = I_0 a B_0 e^{-t/\tau} = \boxed{\frac{v_0 a^2 B_0^2}{R} e^{-t/RC}}$$

Note that the required force decays down to zero! This is in contrast to the capacitorless slidewire circuit, which requires a constant force for all time. It is helpful to analyze this problem in terms of work and energy.

Without a capacitor, the steady force and power required is

$$F_0 = \frac{v_0 a^2 B_0^2}{R} \quad \text{and} \quad P_0 = F_0 v_0 = \frac{v_0^2 A^2 B_0^2}{R} = \frac{\mathcal{E}_m^2}{R} = I_0^2 R.$$

Therefore, all the mechanical power provided by pulling on the slidewire is dissipated in the resistor, lost as thermal energy. The current remains steady, so the necessary force also remains steady.

With a capacitor, a portion of the mechanical power goes toward dissipation in the resistor, but the remaining portion goes toward charging the capacitor. Moreover, as the capacitor's potential difference approaches the value of the emf, current flow through the circuit slows. Therefore, the necessary force decays toward zero.

What fraction of the total mechanical power provided goes into charging the capacitor?

$$\frac{\text{Final capacitor energy}}{\text{Total mechanical work}} = \frac{C\mathcal{E}_m^2/2}{\int_0^\infty P(t)\,dt} = \frac{RC}{2}\left[\int_0^\infty e^{-t/RC}\,dt\right]^{-1} = \frac{1}{2}$$

■

8) (10 points) A semipermeable membrane allows an ion species of positive charge q to freely diffuse into and out of a cell. It is found that the ion's extracellular concentration c_{out} is greater than its intracellular concentration c_{in}.

a) The Nernst potential describes the electric field needed to balance diffusion. First, recalling that $\ln x > 0$ for $x > 1$, mathematically determine the sign of $V_{\text{Nernst}} = (k_B T/q)\ln(c_{\text{out}}/c_{\text{in}})$. Then, using physical reasoning, briefly explain why this is so. You may include a sketch in your explanation.

b) At a lower temperature, how does the magnitude of the Nernst potential change? Again, do this first mathematically and then explain why this is so.

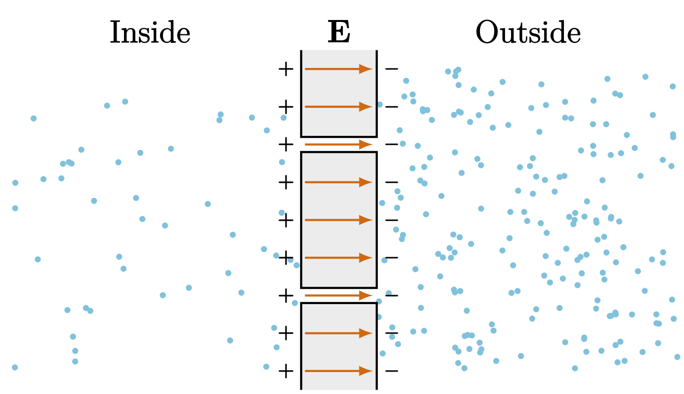
c) You measure the actual cell membrane potential V_{mem} and find that it does not agree with the V_{Nernst} of this ion species, so that there is net diffusion of the ion. Speculate about what might be happening.

Solution

a) To determine the sign of the Nernst potential, we look at the signs of its factors:

- The Boltzmann constant k_B is positive.
- The temperature T is the *absolute* temperature, so it is positive.
- The ion's charge q was assumed to be positive.
- Because $c_{\text{out}} > c_{\text{in}}$, then $c_{\text{out}}/c_{\text{in}} > 1$ so that $\ln(c_{\text{out}}/c_{\text{in}}) > 0$.

Each of its factors is positive, so V_{Nernst} itself is positive.

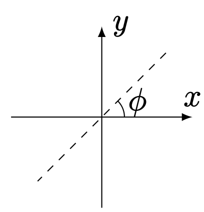


The ions tend to diffuse *inward*, down their concentration gradient. The Nernst potential describes the electric field $\mathbf{E}$ that would be needed to balance diffusion. Because the ions are positively charged, **E** should point outward, so that the ions feel an *outward* electrostatic force. Because the electric potential drops along the direction of the electric field, the electric potential must be higher in the cell's interior. Furthermore, by convention, $V_{\text{out}} = 0$, and we conclude $V_{\text{Nernst}} > 0$.

b) The Nernst potential is *directly proportional* to absolute temperature T. Therefore, the magnitude of the Nernst potential decreases with temperature. Physically, at lower temperature, the microscopic motion of the ions slows. The process of diffusion, where the ions randomly approach and pass through the pores from either side, also slows. Therefore, we don't need as large of an electric field and its concomittant potential difference to balance diffusion at lower temperature.

c) As we have assumed the ions can freely diffuse across the membrane, the actual electric field is *not* balancing the ions' diffusion. We say that the ion concentrations are "off-equilibrium," which typically would need an additional *active* participant to maintain. In lecture, we discussed one possibility being an *ion-exchange pump*, which is responsible for at least half of the ATP consumed in neurons. Note that maintaining these off-equilibrium concentrations is critical to keeping a resting neuron in a state ready for firing. ∎

9) (10 points) Take ϕ to be the angular position of a polarizing filter's transmission axis, as shown in the xy plane. Collimated *unpolarized* light of intensity I_0 propagates along the z-axis. Find a sequence of polarizing filters of varying angular positions $(\phi_1, \phi_2, \ldots, \phi_N)$ that will yield horizontally polarized light ($\phi_N = 0°$) of intensity $I_0/8$. You may use however many polarizers you need.



Solution After passing through the first polarizer, the intensity is $I_0/2$, regardless of polarizer angle. The transmitted intensity following any subsequent polarizer is equal to the product of the incident intensity and $\cos^2\theta$, where θ is the angle *between* the filter's transmission axis and the incident light's polarization. Also, $\phi = 0°$ and $\phi = 180°$ describe the same filter. Note that $\cos^2 45° = 1/2$ and $\cos^2 60° = 1/4$.

Possible solutions include:

$$(60°, 0°) \rightarrow \left(\frac{I_0}{2}, \frac{I_0}{2}\cos^2 60° = \frac{I_0}{8}\right)$$
$$(90°, 45°, 0°) \rightarrow \left(\frac{I_0}{2}, \frac{I_0}{2}\cos^2 45° = \frac{I_0}{4}, \frac{I_0}{4}\cos^2 45° = \frac{I_0}{8}\right)$$
$$(90°, 135°, 180°) \rightarrow \left(\frac{I_0}{2}, \frac{I_0}{2}\cos^2 45° = \frac{I_0}{4}, \frac{I_0}{4}\cos^2 45° = \frac{I_0}{8}\right)$$

∎

10) (10 points) Consider a photoelectric effect apparatus. We find that, for light of wavelength λ_0, the stopping potential is V_0. Suppose we then cut the wavelength in half. In terms of λ_0 and V_0, what is the new stopping potential?

Solution Using $c = f\lambda$, we express the photon's energy in terms of wavelength instead of frequency, $E_\gamma = hf = hc/\lambda$. Before and after changing wavelengths, $\lambda_1 = \lambda_0/2$, we have

$$eV_0 = \frac{hc}{\lambda_0} - \Phi_0 \quad \text{and} \quad eV_1 = \frac{hc}{\lambda_1} - \Phi_0.$$

The metal's work function is unknown, so let's subtract the two equations to eliminate it,

$$V_1 - V_0 = \frac{hc}{e}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_0}\right) = \frac{hc}{e\lambda_0}(2-1) \implies \boxed{V_1 = V_0 + \frac{hc}{e\lambda_0}}$$

When we cut the wavelength in half, each photon's energy increases by an amount hc/λ_0. Subsequently, the maximum kinetic energy of ejected electrons increases by the same amount, for which the additional stopping potential exactly compensates. ■