Solutions

1) (10 points) Four charges are arranged at the corners of a square of side length $a\sqrt{2}$, as shown. Find the magnitude of the electric field at the center of the square, taking $q > 0$.

Solution In RP2-09, we saw that a good choice of coordinate system may provide a significantly simplified solution. In this problem, let’s locate the charges along the $x$ and $y$ axes,

Then, we avoid a lot of cosines and each field component depends only on two charges,

$$E_x = \frac{kq}{a^2} - \frac{4kq}{a^2} = -\frac{3kq}{a^2} \quad \text{and} \quad E_y = \frac{6kq}{a^2} - \frac{2kq}{a^2} = \frac{4kq}{a^2}.$$

So that

$$|E| = \sqrt{E_x^2 + E_y^2} = \frac{kq}{a^2} \sqrt{3^2 + 4^2} = \frac{5kq}{a^2}.$$

2) (10 points) A particle of positive charge $q$ and kinetic energy $K$ is traveling horizontally when it enters a region between two horizontal parallel plates, as shown. The plates, of length $L$, are separated by a gap $d$ and have a potential difference $\Delta V$. Upon exiting the gap, the particle makes an angle $\theta$ with respect to horizontal. Find an expression for $\theta$. 

$$|E| = \sqrt{E_x^2 + E_y^2} = \frac{kq}{a^2} \sqrt{3^2 + 4^2} = \frac{5kq}{a^2}.$$
**Solution** There is no force in the horizontal direction, so the particle’s horizontal velocity, $v_x$, remains constant throughout the problem. The duration $\Delta t$ that the particle spends between the plates is found from

$$L = v_x \Delta t \implies \Delta t = \frac{L}{v_x}.$$  

In the gap, the particle feels a force of magnitude $qE = q\Delta V/d$ upwards. Therefore, the particle’s upward acceleration is $a_y = q\Delta V/(md)$. The particle’s vertical velocity, after it exits the gap, is

$$v_y = a_y \Delta t = \left(\frac{q\Delta V}{md}\right) \left(\frac{L}{v_x}\right)$$

Then,

$$\tan \theta = \frac{v_y}{v_x} = \frac{qL\Delta V}{dmc_y^2} \implies \theta = \arctan\left(\frac{qL\Delta V}{2dK}\right)$$

where we have used $K = \frac{mv_x^2}{2}$.

3) (10 points) In the circuit below, $\mathcal{E}_0 = 3$ V, $C_1 = 6$ µF, $C_2 = 2$ µF, and $C_3 = 1$ µF. Find each capacitor’s potential difference, charge, and energy.

**Solution** We can use the tabular method presented in the Week 3 lectures. Capacitors $C_2$ and $C_3$ are in parallel, so we replace them with capacitor $C_4$. Then, capacitors $C_1$ and $C_4$ are in series, so we replace them with capacitor $C_5$.

$$\begin{array}{c}
\mathcal{E}_0 \\
| \\
C_1 & C_2 \\
| \\
\mathcal{E}_0 \\
| \\
C_3
\end{array} \rightarrow
\begin{array}{c}
\mathcal{E}_0 \\
| \\
C_1 \\
| \\
C_4 \\
| \\
\mathcal{E}_0 \\
| \\
C_5
\end{array}$$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$Q_i$ (µC)</th>
<th>$V_i$ (V)</th>
<th>$C_i$ (µF)</th>
<th>$U_i$ (µJ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>1</td>
<td>6</td>
<td>3</td>
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<tr>
<td>2</td>
<td>4</td>
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<td>4</td>
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<td>6</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>3</td>
<td>2</td>
<td>9</td>
</tr>
</tbody>
</table>

Steps I took in filling out the table:

1. $C_1$, $C_2$, and $C_3$ are given.
2. $C_4 = C_2 + C_3$
3. $C_5 = [(1/C_1) + (1/C_4)]^{-1}$
4. $V_5 = \mathcal{E}_0 = 3$ V
5. $Q_5 = V_5 C_5$. In a row, whenever we see that two out of three $Q$, $V$, or $C$ columns are determined, we can find the third value via $Q = VC$.

6. $Q_1 = Q_4 = Q_5$. Series capacitors carry the same charge as their equivalent capacitor.

7. Determine $V_1$ and $V_4$ via $Q = VC$.

8. $V_2 = V_3 = V_4$. Parallel capacitors have the same potential difference as their equivalent capacitor.

9. Determine $Q_2$ and $Q_3$ via $Q = VC$.

10. $U_i = Q_i V_i / 2$

To check our work, we can confirm that $U_1 + U_2 + U_3 = U_1 + U_4 = U_5$.

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4) (10 points) Consider a spherical conducting shell of resistivity $\rho$, radius $r_1$, and shell thickness $d_1$. When we apply a potential difference between the shell’s inner and outer surfaces, a radial current flows. 

a) Assuming a thin shell thickness $d_1 \ll r_1$, find the electrical resistance $R_1$ to radial current flow through the shell.

b) We uniformly stretch the sphere out to a larger radius $r_2$ with a thinner shell thickness $d_2$ so that the new resistance is $R_2$. Find an expression for $R_2 / R_1$ in terms of $r_2$ and $r_1$.

Solution

a) With current flowing radially, the spherical shell acts as a wire of short length, $L = d_1$, and large cross section area, $A = 4\pi r_1^2$,

$$R_1 = \frac{\rho L}{A} = \frac{\rho d_1}{4\pi r_1^2}$$

b) The mass of the spherical shell doesn’t change when we stretch it out. Therefore, the volume of the spherical shell doesn’t change,

$$V_1 = V_2 \implies 4\pi r_1^2 d_1 = 4\pi r_2^2 d_2 \implies \frac{d_2}{d_1} = \frac{r_2^2}{r_1^2},$$

where we have made our usual approximation for the volume of a thin spherical shell. Then,

$$\frac{R_2}{R_1} = \frac{\rho d_2}{4\pi r_2^2} \cdot \frac{4\pi r_1^2}{\rho d_1} = \frac{d_2}{d_1} \cdot \frac{r_1^2}{r_2^2} = \left(\frac{r_1^2}{r_2^2}\right)$$

For example, if we double the radius, $r_2 = 2r_1$, then the shell’s resistance decreases by a factor of 16. The reduction in wire length decreases the resistance by a factor of 4, and the increase in wire cross-section decreases the resistance by another factor of 4.

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5) (10 points) After installing the battery in the circuit below, we wait a long time for the capacitors to fully charge. Find the the fully charged potential difference across each capacitor.

\[ \begin{array}{c}
E_0 \\
\hline
R_0 & \text{2} & \text{R}_0 & \text{2} \times C_0 \\
\hline
\end{array} \]

Solution  The key insight is that, after the capacitors are fully charged, no more current flows into the capacitor branch. Then, the same current $I$ flows through the battery and both resistors.
Traversing the left loop clockwise,

\[ \mathcal{E}_0 - IR_0 - 2IR_0 \implies I = \frac{\mathcal{E}_0}{3R_0} \]

The potential across the \( 2R_0 \) resistor is then

\[ V_C = I \cdot 2R_0 = \frac{\mathcal{E}_0}{3R_0} \cdot 2R_0 = \frac{2}{3}\mathcal{E}_0. \]

Therefore, the total potential difference across the two capacitors, connected in series, is also equal to \( 2\mathcal{E}_0/3 \), which we may diagram as

\[
\begin{array}{c}
\text{2E}_0 \\
\hline
2C_0 \\
\hline
\text{3E}_0 \\
\hline
C_0 \\
\hline
C_1
\end{array}
\]

The equivalent capacitance \( C_1 \) is found as

\[
C_1 = \left( \frac{1}{C_0} + \frac{1}{2C_0} \right)^{-1} = \left( \frac{3}{2C_0} \right)^{-1} = \frac{2}{3}C_0.
\]

The charge on \( C_1 \) is then

\[ Q_1 = V_1C_1 = \frac{2\mathcal{E}_0}{3} \cdot \frac{2C_0}{3} = \frac{4\mathcal{E}_0C_0}{9} \]

The charges on capacitors in series are equal to charge on their equivalent capacitor. Therefore,

\[ V_{C_0} = \frac{Q_1}{C_0} = \frac{4\mathcal{E}_0}{9} \quad \text{and} \quad V_{2C_0} = \frac{Q_1}{2C_0} = \frac{2\mathcal{E}_0}{9} \]

\[ 6) \quad (10 \text{ points}) \quad \text{A current 4}I_0 \text{ flows along the z-axis and a current 3}I_0 \text{ flows along the y-axis, as shown. Find the magnitude of the resulting magnetic field at point } P. \]

\[ \text{Solution} \quad \text{At point } P, \text{ a distance } a \text{ from the origin, the components of the magnetic field are} \]

\[ B_y = \frac{\mu_0 4I_0}{2\pi a} \quad \text{and} \quad B_z = -\frac{\mu_0 3I_0}{2\pi a}. \]

Hence,

\[ B = \sqrt{B_y^2 + B_z^2} = \frac{\mu_0 I_0}{2\pi a} \sqrt{4^2 + 3^2} = \frac{5\mu_0 I_0}{2\pi a} \]
7) (10 points) A slidewire is in nonuniform magnetic field \( |B| = \alpha x \), where \( \alpha \) is a positive constant. You pull on the slidewire with constant velocity \( v \), as shown. a) Find the magnitude and direction of the resulting current. b) Find the magnetic drag force and the mechanical power your hand must provide to maintain the velocity.

![Diagram of a slidewire in a magnetic field](image)

**Solution**

a) Consider the slidewire beginning at time \( t \) and ending at \( t + \Delta t \), a short time later. During this time interval, the slidewire travels a distance \( \Delta x = v \Delta t \). The magnitude of the flux through the loop increases by

\[
|\Delta \Phi| = B(x) \Delta A = (\alpha vt)(a \Delta x) = (\alpha vt)(av\Delta t) = \alpha v^2 ta \Delta t
\]

Then, the magnitude of the emf and the resulting current are

\[
\mathcal{E} = \frac{|\Delta \Phi|}{\Delta t} = \alpha v^2 ta \quad \text{and} \quad I = \frac{\mathcal{E}}{R} = \frac{\alpha v^2 ta}{R}.
\]

Interesting — the current increases linearly with time. The magnetic flux through the loop is increasing *out of the page*. Therefore, by Lenz’s law, the induced current is clockwise to compensate for the lost flux.

b) The magnetic drag force and mechanical power are

\[
F_d = IaB = \frac{\alpha v^2 ta^2}{R} \cdot (\alpha vt) = \frac{\alpha^2 v^3 t^2 a^2}{R} \quad \text{and} \quad P = F_d v = \frac{\alpha^2 v^4 t^2 a^2}{R}
\]

To check your work, you can confirm that \( P = I^2 R \), showing that all the power dissipated in the resistor is supplied by the force pulling on the slidewire.

8) (10 points) A semipermeable membrane of thickness \( d \) allows an ion species of positive charge \( q \) to freely diffuse into and out of a cell while preventing all other species from passing. At temperature \( T \), the ion concentrations are in equilibrium, and it is noted that, inside the membrane, there is a strong electric field \( E_0 \) directed inward, into the cell. a) Find an expression for \( c_{\text{out}}/c_{\text{in}} \) in terms of the given quantities. b) Is the ion concentration larger inside or outside the cell? Using physical reasoning, including a sketch, briefly explain why this is so.

**Solution**

a) By convention, the electric potential outside the cell is zero. The electric field points inward, so that the potential inside the cell, equal to the membrane potential, is negative,

\[
V_{\text{mem}} = -E_0 d
\]
We are told that the ion concentrations are in equilibrium. Therefore, the Nernst potential for the ion species is equal to the membrane potential,

$$V_{\text{Nernst}} = V_{\text{mem}} \implies \frac{kT}{q} \ln \left( \frac{c_{\text{out}}}{c_{\text{in}}} \right) = -E_0d \implies \frac{c_{\text{out}}}{c_{\text{in}}} = e^{-qE_0d/(kT)}$$

b) Because $q > 0$, our result gives $c_{\text{out}}/c_{\text{in}} < 1$, so that the ion concentration is larger inside the cell. It is very easy to get confused in these problems and be off by a sign. Therefore, we should back our mathematical result with a little physical reasoning.

We are told that the electric field points inward and that the ion species of interest has positive charge. Our calculation showed that, at equilibrium, the inside ion concentration is higher than the outside ion concentration, as shown above. The essential basis for equilibrium may be stated: “The tendency for the ions to diffuse out of the cell, down their concentration gradient, is balanced by the tendency for the ions to accelerate back into the cell, along the membrane’s electric field.”

9) (10 points) In a photoelectric demonstration lab, the maximum kinetic energy of photoelectrons is $K_0$. Reducing the wavelength of the incident light to half of its initial value increases the maximum photoelectron kinetic energy to $K_1$. a) What is the work function of the cathode? b) What was the initial wavelength?

Solution This is essentially the same as RPF-18 (MP09-Q16).

a) The work-energy relations for both wavelengths are

$$\Phi_0 + K_0 = \frac{hc}{\lambda_0} \quad \text{and} \quad \Phi_0 + K_1 = \frac{hc}{\lambda_1}$$

Using $\lambda_1 = \lambda_0/2$, we can solve for $\Phi_0$,

$$2(\Phi_0 + K_0) = \Phi_0 + K_1 \implies \Phi_0 = K_1 - 2K_0$$

b)

$$\lambda_0 = \frac{hc}{\Phi_0 + K_0} = \frac{hc}{K_1 - K_0}$$

10) (10 points) A particle of mass $m$ is in a box of length $L$. The system is in contact with a heat reservoir at temperature $T$. At what value of $T$ is the particle exactly twice as likely to be in the ground state rather than in the first excited state?
\textbf{Solution}  At temperature $T$, the Boltzmann distribution gives the relative occupation probabilities of the system’s energy states. A particle-in-a-box’s ground state corresponds to $n = 1$ and its first excited state corresponds to $n = 2$. Then,

$$
\frac{P_2}{P_1} = \frac{e^{-E_2/(kT)}}{e^{-E_1/(kT)}} = e^{-\Delta E/(kT)}
$$

where

$$
\Delta E = E_2 - E_1 = \frac{\hbar^2}{8mL^2}(2^2 - 1^2) = \frac{3\hbar^2}{8mL^2}.
$$

Setting $P_1 = 2P_2$,

$$
e^{-\Delta E/(kT)} = \frac{1}{2} \implies -\frac{\Delta E}{kT} = \ln \left(\frac{1}{2}\right) \implies \frac{\Delta E}{kT} = \ln 2 \implies T = \frac{\Delta E}{k \ln 2} = \frac{3\hbar^2}{8mL^2 k \ln 2}
$$

The occupation probabilities for the particle-in-a-box’s first four energy states, at temperature $T$, are shown below.