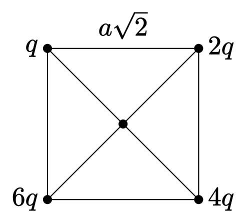
Physics 5C - Final

Wednesday, June 12, 3-6PM

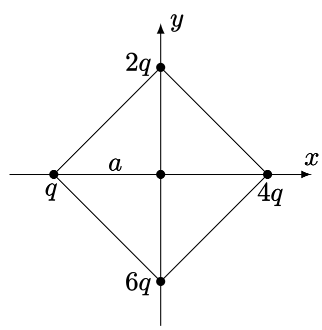
UCLA / Spring 2024 / Brian Naranjo

Solutions

1) (10 points) Four charges are arranged at the corners of a square of side length $a\sqrt{2}$, as shown. Find the *magnitude* of the electric field at the center of the square, taking $q > 0$.



Solution In RP2-09, we saw that a good choice of coordinate system may provide a significantly simplified solution. In this problem, let's locate the charges along the x and y axes,



Then, we avoid a lot of cosines and each field component depends only on two charges,

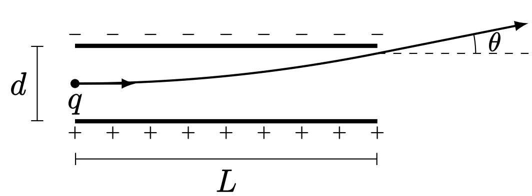
$$E_x = \frac{kq}{a^2} - \frac{4kq}{a^2} = -\frac{3kq}{a^2} \quad \text{and} \quad E_y = \frac{6kq}{a^2} - \frac{2kq}{a^2} = \frac{4kq}{a^2}.$$

So that

$$|E| = \sqrt{E_x^2 + E_y^2} = \frac{kq}{a^2}\sqrt{3^2 + 4^2} = \boxed{\frac{5kq}{a^2}}$$

■

2) (10 points) A particle of positive charge q and kinetic energy K is traveling horizontally when it enters a region between two horizontal parallel plates, as shown. The plates, of length L, are separated by a gap d and have a potential difference ΔV. Upon exiting the gap, the particle makes an angle θ with respect to horizontal. Find an expression for θ.



Solution There is no force in the horizontal direction, so the particle's horizontal velocity, v_x, remains constant throughout the problem. The duration Δt that the particle spends between the plates is found from

$$L = v_x \Delta t \implies \Delta t = \frac{L}{v_x}.$$

In the gap, the particle feels a force of magnitude $qE = q\Delta V/d$ upwards. Therefore, the particle's upward acceleration is $a_y = q\Delta V/(md)$. The particle's vertical velocity, after it exits the gap, is

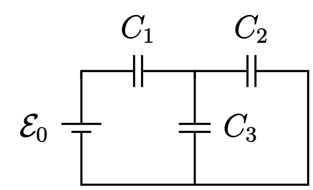
$$v_y = a_y \Delta t = \left(\frac{q\Delta V}{md}\right)\left(\frac{L}{v_x}\right)$$

Then,

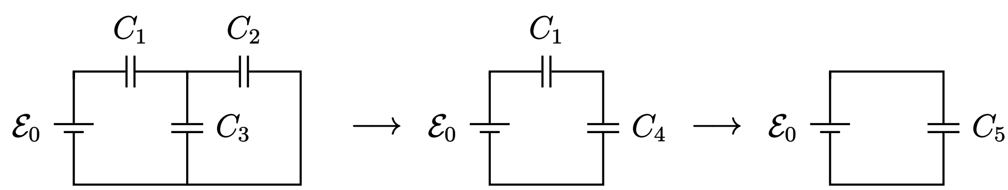
$$\tan\theta = \frac{v_y}{v_x} = \frac{qL\Delta V}{dmv_x^2} \implies \boxed{\theta = \arctan\left(\frac{qL\Delta V}{2dK}\right)}$$

where we have used $K = mv_x^2/2$. ■

3) (10 points) In the circuit below, $\mathcal{E}_0 = 3\ \text{V}, C_1 = 6\ \mu\text{F}, C_2 = 2\ \mu\text{F}$, and $C_3 = 1\ \mu\text{F}$. Find each capacitor's potential difference, charge, and energy.



Solution We can use the tabular method presented in the Week 3 lectures. Capacitors C_2 and C_3 are in parallel, so we replace them with capacitor C_4. Then, capacitors C_1 and C_4 are in series, so we replace them with capacitor C_5.



i	Q_i (μC)	V_i (V)	C_i (μF)	U_i (μJ)
1	6	1	6	3
2	4	2	2	4
3	2	2	1	2
4	6	2	3	6
5	6	3	2	9

Steps I took in filling out the table:

1. C_1, C_2, and C_3 are given.
2. $C_4 = C_2 + C_3$
3. $C_5 = [(1/C_1) + (1/C_4)]^{-1}$
4. $V_5 = \mathcal{E}_0 = 3$ V

5. $Q_5 = V_5 C_5$. In a row, whenever we see that two out of three Q, V, or C columns are determined, we can find the third value via $Q = VC$.
6. $Q_1 = Q_4 = Q_5$. Series capacitors carry the same charge as their equivalent capacitor.
7. Determine V_1 and V_4 via $Q = VC$.
8. $V_2 = V_3 = V_4$. Parallel capacitors have the same potential difference as their equivalent capacitor.
9. Determine Q_2 and Q_3 via $Q = VC$.
10. $U_i = Q_i V_i / 2$

To check our work, we can confirm that $U_1 + U_2 + U_3 = U_1 + U_4 = U_5$. ■

4) (10 points) Consider a spherical conducting shell of resistivity ρ, radius r_1, and shell thickness d_1. When we apply a potential difference between the shell's inner and outer surfaces, a radial current flows. **a)** Assuming a thin shell thickness $d_1 \ll r_1$, find the electrical resistance R_1 to radial current flow through the shell. **b)** We uniformly stretch the sphere out to a larger radius r_2 with a thinner shell thickness d_2 so that the new resistance is R_2. Find an expression for R_2/R_1 in terms of r_2 and r_1.

Solution

a) With current flowing radially, the spherical shell acts as a wire of short length, $L = d_1$, and large cross section area, $A = 4\pi r_1^2$,

$$\boxed{R_1 = \frac{\rho L}{A} = \frac{\rho d_1}{4\pi r_1^2}}$$

b) The mass of the spherical shell doesn't change when we stretch it out. Therefore, the volume of the spherical shell doesn't change,

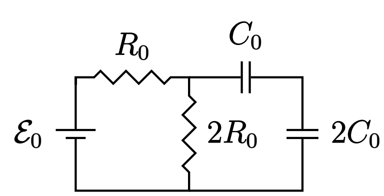
$$V_1 = V_2 \implies 4\pi r_1^2 d_1 = 4\pi r_2^2 d_2 \implies \frac{d_2}{d_1} = \frac{r_1^2}{r_2^2},$$

where we have made our usual approximation for the volume of a thin spherical shell. Then,

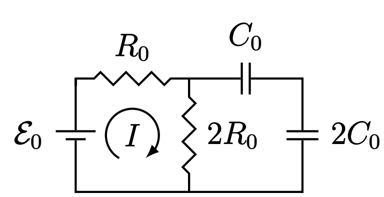
$$\frac{R_2}{R_1} = \frac{\rho d_2}{4\pi r_2^2} \cdot \frac{4\pi r_1^2}{\rho d_1} = \frac{d_2}{d_1} \cdot \frac{r_1^2}{r_2^2} = \boxed{\frac{r_1^4}{r_2^4}}$$

For example, if we double the radius, $r_2 = 2r_1$, then the shell's resistance decreases by a factor of 16. The reduction in wire length decreases the resistance by a factor of 4, and the increase in wire cross-section decreases the resistance by another factor of 4. ■

5) (10 points) After installing the battery in the circuit below, we wait a long time for the capacitors to fully charge. Find the the fully charged potential difference across each capacitor.



Solution The key insight is that, after the capacitors are fully charged, no more current flows into the capacitor branch. Then, the same current I flows through the battery and both resistors.



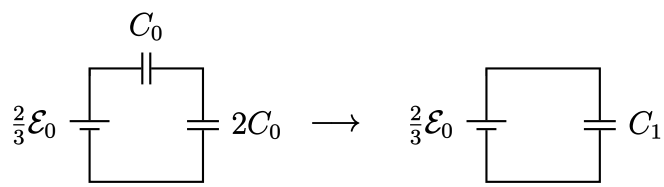
Traversing the left loop clockwise,

$$\mathcal{E}_0 - IR_0 - I2R_0 \implies I = \frac{\mathcal{E}_0}{3R_0}$$

The potential across the $2R_0$ resistor is then

$$V_C = I \cdot 2R_0 = \frac{\mathcal{E}_0}{3R_0} \cdot 2R_0 = \frac{2}{3}\mathcal{E}_0.$$

Therefore, the total potential difference across the two capacitors, connected in series, is also equal to $2\mathcal{E}_0/3$, which we may diagram as



The equivalent capacitance C_1 is found as

$$C_1 = \left(\frac{1}{C_0} + \frac{1}{2C_0}\right)^{-1} = \left(\frac{3}{2C_0}\right)^{-1} = \frac{2}{3}C_0.$$

The charge on C_1 is then

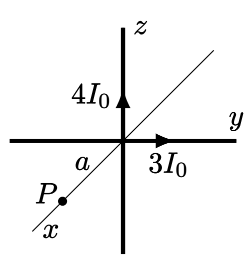
$$Q_1 = V_1 C_1 = \frac{2\mathcal{E}_0}{3} \cdot \frac{2C_0}{3} = \frac{4}{9}\mathcal{E}_0 C_0$$

The charges on capacitors in series are equal to charge on their equivalent capacitor. Therefore,

$$V_{C_0} = \frac{Q_1}{C_0} = \boxed{\frac{4}{9}\mathcal{E}_0} \quad \text{and} \quad V_{2C_0} = \frac{Q_1}{2C_0} = \boxed{\frac{2}{9}\mathcal{E}_0}$$

■

6) (10 points) A current $4I_0$ flows along the z-axis and a current $3I_0$ flows along the y-axis, as shown. Find the *magnitude* of the resulting magnetic field at point P.



Solution At point P, a distance a from the origin, the components of the magnetic field are

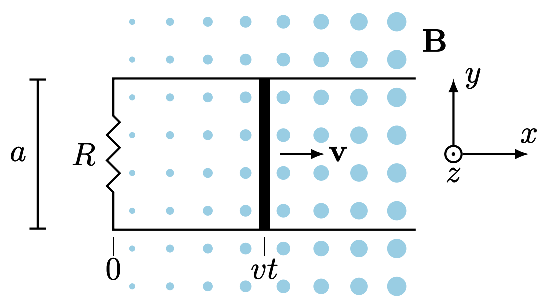
$$B_y = \frac{\mu_0 4I_0}{2\pi a} \quad \text{and} \quad B_z = -\frac{\mu_0 3I_0}{2\pi a}.$$

Hence,

$$B = \sqrt{B_y^2 + B_z^2} = \frac{\mu_0 I_0}{2\pi a}\sqrt{4^2 + 3^2} = \boxed{\frac{5\mu_0 I_0}{2\pi a}}$$

■

7) (10 points) A slidewire is in nonuniform magnetic field $|\mathbf{B}| = \alpha x$, where α is a positive constant. You pull on the slidewire with constant velocity v, as shown. **a)** Find the magnitude and direction of the resulting current. **b)** Find the magnetic drag force and the mechanical power your hand must provide to maintain the velocity.



Solution

a) Consider the slidewire beginning at time t and ending at $t + \Delta t$, a short time later. During this time interval, the slidewire travels a distance $\Delta x = v\Delta t$. The magnitude of the flux through the loop increases by

$$|\Delta\Phi| = B(x)\Delta A = (\alpha vt)(a\Delta x) = (\alpha vt)(av\Delta t) = \alpha v^2 ta\Delta t$$

Then, the magnitude of the emf and the resulting current are

$$\mathcal{E} = \left|\frac{\Delta\Phi}{\Delta t}\right| = \alpha v^2 ta \quad \text{and} \quad I = \frac{\mathcal{E}}{R} = \boxed{\frac{\alpha v^2 ta}{R}}.$$

Interesting — the current increases linearly with time. The magnetic flux through the loop is increasing *out of the page*. Therefore, by Lenz's law, the induced current is $\boxed{\text{clockwise}}$ to compensate for the lost flux.

b) The magnetic drag force and mechanical power are

$$F_d = IaB = \frac{\alpha v^2 ta^2}{R} \cdot (\alpha vt) = \boxed{\frac{\alpha^2 v^3 t^2 a^2}{R}} \quad \text{and} \quad P = F_d v = \boxed{\frac{\alpha^2 v^4 t^2 a^2}{R}}$$

To check your work, you can confirm that $P = I^2 R$, showing that all the power dissipated in the resistor is supplied by the force pulling on the slidewire. ■

8) (10 points) A semipermeable membrane of thickness d allows an ion species of positive charge q to freely diffuse into and out of a cell while preventing all other species from passing. At temperature T, the ion concentrations are in equilibrium, and it is noted that, inside the membrane, there is a strong electric field E_0 directed inward, into the cell. **a)** Find an expression for $c_{\text{out}}/c_{\text{in}}$ in terms of the given quantities. **b)** Is the ion concentration larger inside or outside the cell? Using physical reasoning, including a sketch, briefly explain why this is so.

Solution

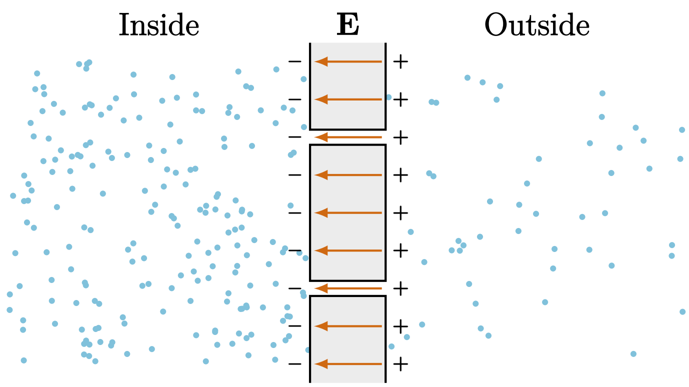
a) By convention, the electric potential outside the cell is zero. The electric field points inward, so that the potential inside the cell, equal to the membrane potential, is negative,

$$V_{\text{mem}} = -E_0 d$$

We are told that the ion concentrations are in equilibrium. Therefore, the Nernst potential for the ion species is equal to the membrane potential,

$$V_{\text{Nernst}} = V_{\text{mem}} \implies \frac{kT}{q} \ln\left(\frac{c_{\text{out}}}{c_{\text{in}}}\right) = -E_0 d \implies \boxed{\frac{c_{\text{out}}}{c_{\text{in}}} = e^{-qE_0 d/(kT)}}$$

b) Because $q > 0$, our result gives $c_{\text{out}}/c_{\text{in}} < 1$, so that the ion concentration is larger $\boxed{\text{inside}}$ the cell. It is very easy to get confused in these problems and be off by a sign. Therefore, we should back our mathematical result with a little physical reasoning.



We are told that the electric field points inward and that the ion species of interest has positive charge. Our calculation showed that, at equilibrium, the inside ion concentration is higher than the outside ion concentration, as shown above. The essential basis for equilibrium may be stated: "The tendency for the ions to diffuse out of the cell, down their concentration gradient, is balanced by the tendency for the ions to accelerate back into the cell, along the membrane's electric field." ■

9) (10 points) In a photoelectric demonstration lab, the maximum kinetic energy of photoelectrons is K_0. Reducing the wavelength of the incident light to half of its initial value increases the maximum photoelectron kinetic energy to K_1. **a)** What is the work function of the cathode? **b)** What was the initial wavelength?

Solution This is essentially the same as RPF-18 (MP09-Q16).
a) The work-energy relations for both wavelengths are

$$\Phi_0 + K_0 = \frac{hc}{\lambda_0} \quad \text{and} \quad \Phi_0 + K_1 = \frac{hc}{\lambda_1}$$

Using $\lambda_1 = \lambda_0/2$, we can solve for Φ_0,

$$2(\Phi_0 + K_0) = \Phi_0 + K_1 \implies \boxed{\Phi_0 = K_1 - 2K_0}$$

b)

$$\lambda_0 = \frac{hc}{\Phi_0 + K_0} = \boxed{\frac{hc}{K_1 - K_0}}$$

■

10) (10 points) A particle of mass m is in a box of length L. The system is in contact with a heat reservoir at temperature T. At what value of T is the particle exactly twice as likely to be in the ground state rather than in the first excited state?

Solution At temperature T, the Boltzmann distribution gives the relative occupation probabilities of the system's energy states. A particle-in-a-box's ground state corresponds to $n = 1$ and its first excited state corresponds to $n = 2$. Then,

$$\frac{P_2}{P_1} = \frac{e^{-E_2/(kT)}}{e^{-E_1/(kT)}} = e^{-\Delta E/(kT)}$$

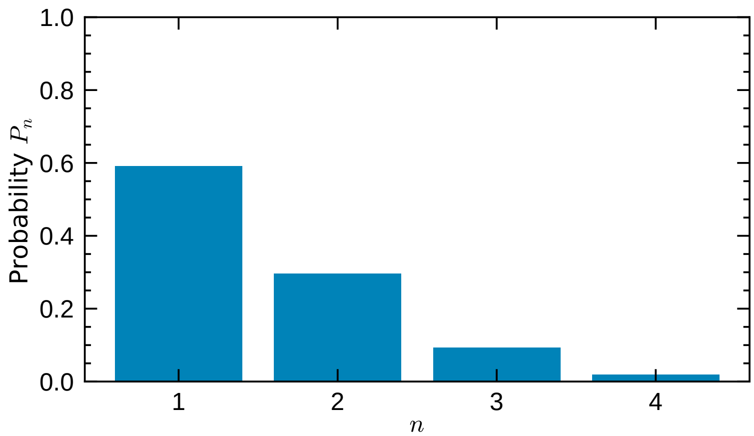
where

$$\Delta E = E_2 - E_1 = \frac{h^2}{8mL^2}(2^2 - 1^2) = \frac{3h^2}{8mL^2}.$$

Setting $P_1 = 2P_2$,

$$e^{-\Delta E/(kT)} = \frac{1}{2} \implies -\frac{\Delta E}{kT} = \ln\left(\frac{1}{2}\right) \implies \frac{\Delta E}{kT} = \ln 2 \implies T = \frac{\Delta E}{k \ln 2} = \boxed{\frac{3h^2}{8mL^2 k \ln 2}}$$

The occupation probabilities for the particle-in-a-box's first four energy states, at temperature T, are shown below.



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