Solutions

1) (10 points) A ray of light is incident normally on face $AB$ of a glass prism of refractive index $n$, immersed in air. Find the largest value of the angle $\alpha$ so that the ray is totally internally reflected at face $AC$.

\[ \text{Solution.} \] At the critical angle, the refracted angle is $90^\circ$. Snell’s law gives

\[ n \sin \bar{\alpha} = 1 \implies n \cos \alpha = 1 \implies \alpha = \arccos(1/n) \]

2) (10 points) An object, possibly either virtual or real, is located at $z = 0$. Find the focal length $f$ and location $z_L$ of a lens such that the resulting image, possibly either virtual or real, is located at $z' = d$ with a magnification of $m = +2$.

\[ \text{Solution.} \] As described, the system satisfies the following two equations,

\[ m = -\frac{s'}{s} = 2 \]

\[ d = s + s' \]

Combining these,

\[ d = s - 2s = -s \]

Therefore, $s = -d$ and $s' = 2d$. The object is located a distance $d$ from the lens, and, because $s < 0$, the object is on the transmitted side. So, $z_L = -d$. Lastly, plugging into the thin lens equation then gives us $f$,

\[ \frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \implies \frac{1}{s} - \frac{1}{2s} = \frac{1}{f} \implies \frac{1}{2s} = \frac{1}{f} \implies f = 2s = -2d \]

In summary, $f = -2d$ and $z_L = -d$.

You weren’t asked to draw the ray diagram, but here it is, with the lens’s three principal rays indicated by arrows,
3) (10 points) Two coherent sources $S_1$ and $S_2$ of equal power emit at the same wavelength $\lambda_0$, but they are $180^\circ$ out of phase with each other. Assuming $\lambda_0 \ll d$, find the largest finite value $x > 0$ on the $x$-axis for which there is complete destructive interference. Justify your approximation.

**Solution.** The OPD of the two sources to observation point $P$ on the $x$ axis is

$$\text{OPD} = \sqrt{d^2 + x^2} - x$$
$$= x \sqrt{1 + \left(\frac{d}{x}\right)^2} - x$$
$$\approx x \left[ 1 + \frac{1}{2} \left(\frac{d}{x}\right)^2 \right] - x$$
$$= \frac{d^2}{2x},$$

where, in using the binomial approximation, we have assumed, to be justified later, that $d \ll x$. For destructive interference,

$$\text{OPD} = \frac{d^2}{2x} = m\lambda_0 \text{ m = 0, 1, 2, . . .}$$

The case $m = 0$ corresponds to where, as $x \to \infty$, the two optical paths become equal, giving destructive interference. However, we are asked to find the largest finite value of $x$, which corresponds to $m = 1$, an OPD of one wavelength,

$$x = \frac{d^2}{2\lambda_0}$$

Now, to justify the binomial approximation,

$$\frac{d}{x} = \frac{2\lambda_0}{d} \ll 1$$
4) (15 points) Light of wavelength $\lambda_0$ is incident in air, from above, on a thin film of thickness $t = 2\lambda_0/3$ and width $w$ placed on a substrate of refractive index $n_0 = 1.6$. If the film’s index of refraction varies over its width according to $n(x) = 1 + (x/w)$, find the two values of $x$ for which the reflected light constructively interferes with the incident light.\(^1\)

![Diagram of light incident on thin film](image)

**Solution.**

**Case 1:** $1 < n(x) < n_0 = 1.6 \iff 0 < x < 0.6w$

The beam reflecting off the top of the thin film incurs a 180° phase shift, and the beam reflecting off the bottom of the thin film also incurs a 180° phase shift. Therefore, the condition for the two beams to constructively interfere is

$$\text{OPD} = 2tn(x) = m\lambda_0 \implies n(x) = \frac{3m}{4} (m = 1, 2, 3, \ldots)$$

The only value of $m$ that satisfies the restriction on $n(x)$ is $m = 2$, which gives $n(x) = 1.5$. Therefore, there is constructive interference at $x = w/2 = 0.5w$.

**Case 2:** $n_0 = 1.6 < n(x) < 2 \iff 0.6w < x < w$

The beam reflecting off the top of the thin film incurs a 180° phase shift, but the beam reflecting off the bottom of the thin film does not incur a phase shift. Therefore, the condition for the two beams two constructively interfere is

$$\text{OPD} = 2tn(x) = (m + \frac{1}{2})\lambda_0 \implies n(x) = \frac{3(m + (1/2))}{4} (m = 0, 1, 2, \ldots)$$

The only value of $m$ that satisfies the restriction on $n(x)$ is $m = 2$, which gives $n(x) = 15/8 = 1.875$. Therefore, there is constructive interference at $x = 7w/8 = 0.875w$.

5) (15 points) A current $I_0$ flows along the $z$ axis, in the positive $z$ direction, terminating at the origin, where a positive charge $Q(t) = I_0 t$ accumulates. Loop $C$ is a counter-clockwise circular path of radius $a$, centered on the $z$ axis, in the plane $z = d$, as shown. Assuming $a \ll d$, find the line integral of $\mathbf{B}$ around loop $C$.

\(^1\)Clarification: We want to find the two values of $x$ for which there is complete reflection.
Solution. We are asked to find the line integral of the magnetic field around a closed path, so we should immediately think to use Ampère’s law,

\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}. \]

We take \( C \) to be the given curve. We are free to choose any surface bound by \( C \) as the flux surface. We simply choose the flat disk \( S \) of radius \( a \). No current pierces \( S \), so \( I_{\text{enc}} = 0 \), and we just need to calculate the electric field’s flux through \( S \).

Using the approximation \( d \ll a \), the electric field due to the point charge \( Q \) is approximately constant over surface \( S \), and the electric flux is

\[ \Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a} \approx \left( \frac{Q}{4\pi \epsilon_0 d^2} \right) \pi a^2 = \frac{I_0 \pi a^2}{4\epsilon_0 d^2} \]

Then,

\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I_0 a^2}{4d^2} \]

6) (20 points) A cutaway view of an ideal toroidal coil, whose axis-of-symmetry coincides with the \( z \) axis, is shown below. It has rectangular cross-section, inner radius \( a \), outer radius \( b \), height \( c \), current \( I_0 \), and a total of \( N \) turns. There is an electric field \( E = -E_0 \hat{z} \).

a) Find the Poynting vector \( \mathbf{S} \) everywhere.

b) Calculate the flux of the Poynting vector through the toroid’s inner surface, oriented in the outward direction,\(^2\)

\[ \int_S \mathbf{S} \cdot d\mathbf{a} \]

\(^2\)Clarification: The surface integral is over the coil’s entire surface. You will find that the Poynting vector has different values inside and outside the coil. In your surface integral, use the interior value of the Poynting vector.
Solution.
a) Using Ampère’s law, the magnetic field is
\[ B = \begin{cases} \frac{\mu_0 NI_0}{2\pi r} \hat{\phi} & \text{Inside coil} \\ 0 & \text{Outside coil} \end{cases} \]
Then, the Poynting vector is
\[ S = \frac{E \times B}{\mu_0} = \begin{cases} \frac{E_0 NI_0}{2\pi r} \hat{r} & \text{Inside coil} \\ 0 & \text{Outside coil} \end{cases} \]
b) The flux of the Poynting vector through the top and bottom of the coil are both equal to zero, because \( S \cdot \hat{z} = 0 \). The \textit{outward} flux of the Poynting through the coil’s wall at the outer radius is
\[ S_{\text{outer}} = \left( \frac{E_0 NI_0}{2\pi b} \right) 2\pi bh = E_0 NI_0 h \]
The \textit{outward} flux of the Poynting through the coil’s wall at the inner radius is negative, because \( S \cdot da < 0 \), so that
\[ S_{\text{inner}} = - \left( \frac{E_0 NI_0}{2\pi a} \right) 2\pi ah = -E_0 NI_0 h \]
Therefore, flux leaving the coil at the outer radius is equal to the flux entering coil at the inner radius, and we have
\[ \oint_{S} S \cdot da = 0 \]

7) (20 points) An optical system consists of
- A converging lens of focal length \( f > 0 \), located at \( z = 2f \).
- A concave mirror, with radius of curvature \( R = 2f \) and vertex located at \( z = 0 \). It is oriented to the right, so that its center-of-curvature is at \( z = 2f \).
- An upright arrow of height \( y_1 \), located in the object plane at \( z = 0.5f \).
The arrow only emits light moving toward the mirror, so don’t consider any light that initially moves toward the lens.

a) Calculate the location \( z_2' \) and total magnification \( m \) of the resulting image.

b) Sketch the system, including three principal rays for both the mirror and the lens. The mirror’s third principal ray reflects off its vertex. Identify all objects and images as real or virtual using our usual notation (e.g., RO\(_1\) for the real object of subsystem 1). Remember that something can be both an image and an object.

c) A ray \((y_1, \theta_1)\) passes through the optical system, beginning in the object plane. Find the ray \((y_2, \theta_2)\) in the image plane.

Solution.
a) The final image location is \( z_2' = 3.5f \) and the total magnification is \( m = -1 \)
b) The system is sketched below. The three principal rays of both the mirror and the lens are indicated with arrows. In the paraxial approximation, we extend the rays to the mirror’s tangent plane.

c) The system’s net optical ray transfer matrix that traces rays from the mirror’s object plane to the lens’s image plane is

\[
M = M_2 \cdot M_1 \\
= \begin{pmatrix} m_2 & 0 \\ -1/f_2 & 1/m_2 \end{pmatrix} \begin{pmatrix} m_1 & 0 \\ -1/f_1 & 1/m_1 \end{pmatrix} \\
= \begin{pmatrix} -1/2 & 0 \\ -1/f & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1/f & 1/2 \end{pmatrix} \\
= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Then,

\[
\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = M \cdot \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} -y_1 \\ -\theta_1 \end{pmatrix}
\]