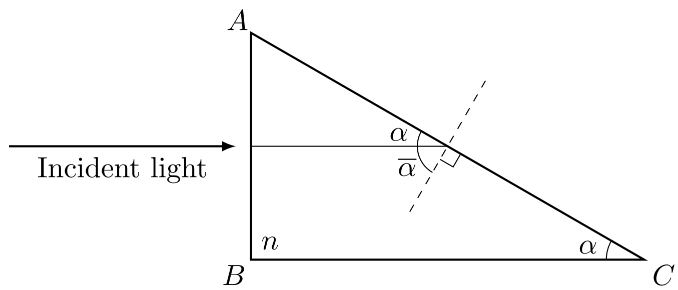
Physics 1C - Second Midterm

Thursday, May 25, 10-11:50AM

UCLA / Spring 2023 / Brian Naranjo

Solutions

1) (10 points) A ray of light is incident normally on face AB of a glass prism of refractive index n, immersed in air. Find the largest value of the angle α so that the ray is totally internally reflected at face AC.



Solution. At the critical angle, the refracted angle is 90°. Snell's law gives

$$n \sin \overline{\alpha} = 1 \implies n \cos \alpha = 1 \implies \boxed{\alpha = \arccos(1/n)}$$

2) (10 points) An object, possibly either virtual or real, is located at $z = 0$. Find the focal length f and location z_L of a lens such that the resulting image, possibly either virtual or real, is located at $z' = d$ with a magnification of $m = +2$.

Solution. As described, the system satisfies the following two equations,

$$m = -\frac{s'}{s} = 2$$
$$d = s + s'$$

Combining these,

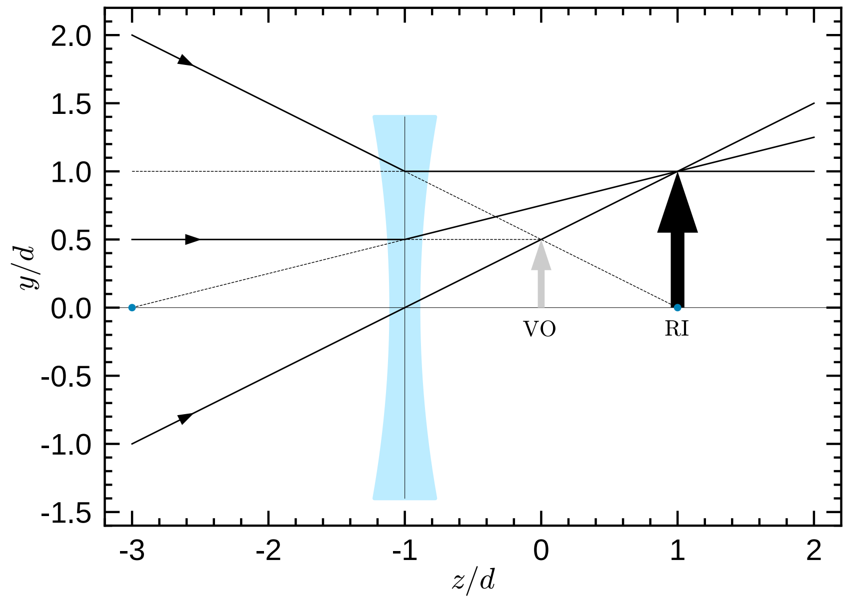
$$d = s - 2s = -s$$

Therefore, $s = -d$ and $s' = 2d$. The object is a located a distance d from the lens, and, because $s < 0$, the object is on the *transmitted* side. So, $z_L = -d$. Lastly, plugging into the thin lens equation then gives us f,

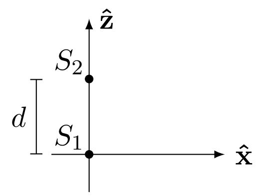
$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \implies \frac{1}{s} - \frac{1}{2s} = \frac{1}{f} \implies \frac{1}{2s} = \frac{1}{f} \implies f = 2s = -2d$$

In summary, $\boxed{f = -2d}$ and $\boxed{z_L = -d}$.

You weren't asked to draw the ray diagram, but here it is, with the lens's three principal rays indicated by arrows,



3) (10 points) Two coherent sources S_1 and S_2 of equal power emit at the same wavelength λ_0, but they are 180° out of phase with each other. Assuming $\lambda_0 \ll d$, find the largest finite value $x > 0$ on the x-axis for which there is complete destructive interference. Justify your approximation.



Solution. The OPD of the two sources to observation point P on the x axis is

$$\begin{aligned}
\text{OPD} &= \sqrt{d^2 + x^2} - x \\
&= x\sqrt{1 + (d/x)^2} - x \\
&\approx x\left[1 + \frac{1}{2}\left(\frac{d}{x}\right)^2\right] - x \\
&= \frac{d^2}{2x},
\end{aligned}$$

where, in using the binomial approximation, we have assumed, to be justified later, that $d \ll x$. For destructive interference,

$$\text{OPD} = \frac{d^2}{2x} = m\lambda_0 \quad m = 0, 1, 2, \ldots$$

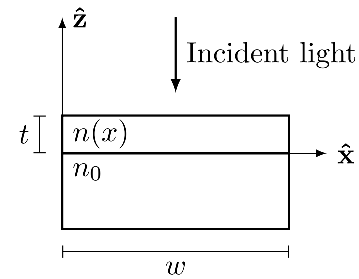
The case $m = 0$ corresponds to where, as $x \to \infty$, the two optical paths become equal, giving destructive interference. However, we are asked to find the largest *finite* value of x, which corresponds to $m = 1$, an OPD of one wavelength,

$$\boxed{x = \frac{d^2}{2\lambda_0}}$$

Now, to justify the binomial approximation,

$$\boxed{\frac{d}{x} = \frac{2\lambda_0}{d} \ll 1}$$

4) (15 points) Light of wavelength λ_0 is incident in air, from above, on a thin film of thickness $t = 2\lambda_0/3$ and width w placed on a substrate of refractive index $n_0 = 1.6$. If the the film's index of refraction varies over its width according to $n(x) = 1 + (x/w)$, find the *two* values of x for which the reflected light *constructively* interferes with the incident light.[1]



Solution.
Case 1: $1 < n(x) < n_0 = 1.6 \iff 0 < x < 0.6\,w$
The beam reflecting off the top of the thin film incurs a 180° phase shift, and the beam reflecting off the bottom of the thin film also incurs a 180° phase shift. Therefore, the condition for the two beams to constructively interfere is

$$\text{OPD} = 2tn(x) = m\lambda_0 \implies n(x) = \frac{3m}{4} \quad (m = 1, 2, 3, \ldots)$$

The only value of m that satisfies the restriction on $n(x)$ is $m = 2$, which gives $n(x) = 1.5$. Therefore, there is constructive interference at $\boxed{x = w/2 = 0.5w}$

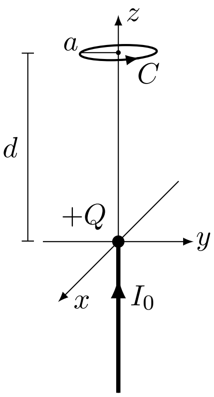
Case 2: $n_0 = 1.6 < n(x) < 2 \iff 0.6\,w < x < w$
The beam reflecting off the top of the thin film incurs a 180° phase shift, but the beam reflecting off the bottom of the thin film does not incur a phase shift. Therefore, the condition for the two beams two constructively interfere is

$$\text{OPD} = 2tn(x) = \left(m + \frac{1}{2}\right)\lambda_0 \implies n(x) = \frac{3[m + (1/2)]}{4} \quad (m = 0, 1, 2, \ldots)$$

The only value of m that satisfies the restriction on $n(x)$ is $m = 2$, which gives $n(x) = 15/8 = 1.875$. Therefore, there is constructive interference at $\boxed{x = 7w/8 = 0.875w}$

5) (15 points) A current I_0 flows along the z axis, in the positive z direction, terminating at the origin, where a positive charge $Q(t) = I_0\,t$ accumulates. Loop C is a counter-clockwise circular path of radius a, centered on the z axis, in the plane $z = d$, as shown. Assuming $a \ll d$, find the line integral of $\mathbf{B}$ around loop C.

[1] Clarification: We want to find the two values of x for which there is complete reflection.



Solution. We are asked to find the line integral of the magnetic field around a closed path, so we should immediately think to use Ampère's law,

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}.$$

We take C to be the given curve. We are free to choose any surface bound by C as the flux surface. We simply choose the flat disk S of radius a. No current pierces S, so $I_{\text{enc}} = 0$, and we just need to calculate the electric field's flux through S.

Using the approximation $d \ll a$, the electric field due to the point charge Q is approximately constant over surface S, and the electric flux is

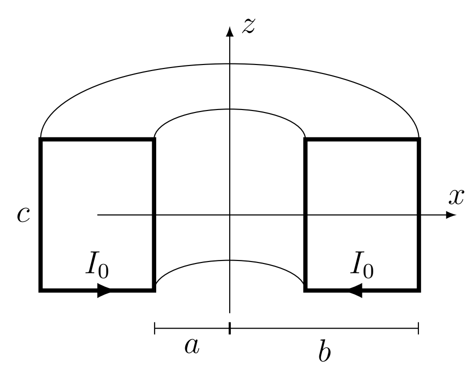
$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{a} \approx \left(\frac{Q}{4\pi\epsilon_0 d^2} \right) \pi a^2 = \frac{I_0 t a^2}{4\epsilon_0 d^2}$$

Then,

$$\boxed{\oint_C \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I_0 a^2}{4d^2}}$$

6) (20 points) A cutaway view of an ideal toroidal coil, whose axis-of-symmetry coincides with the z axis, is shown below. It has rectangular cross-section, inner radius a, outer radius b, height c, current I_0, and a total of N turns. There is an electric field $E = -E_0 \hat{\mathbf{z}}$.
a) Find the Poynting vector $\mathbf{S}$ *everywhere.*
b) Calculate the flux of the Poynting vector through the toroid's inner surface, oriented in the *outward* direction,[2]

$$\oint_S \mathbf{S} \cdot d\mathbf{a}$$



[2] Clarification: The surface integral is over the coil's *entire* surface. You will find that the Poynting vector has different values inside and outside the coil. In your surface integral, use the interior value of the Poynting vector.

Solution.
a) Using Ampère's law, the magnetic field is

$$\mathbf{B} = \begin{cases} \dfrac{\mu_0 N I_0}{2\pi r}\hat{\boldsymbol{\phi}} & \text{Inside coil} \\ 0 & \text{Outside coil} \end{cases}$$

Then, the Poynting vector is

$$\boxed{\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \begin{cases} \dfrac{E_0 N I_0}{2\pi r}\hat{\mathbf{r}} & \text{Inside coil} \\ 0 & \text{Outside coil} \end{cases}}$$

b) The flux of the Poynting vector through the top and bottom of the coil are both equal to zero, because $\mathbf{S}\cdot\hat{\mathbf{z}} = 0$. The *outward* flux of the Poynting through the coil's wall at the outer radius is

$$S_{\text{outer}} = \left(\frac{E_0 N I_0}{2\pi b}\right) 2\pi b h = E_0 N I_0 h$$

The *outward* flux of the Poynting through the coil's wall at the inner radius is negative, because $\mathbf{S}\cdot d\mathbf{a} < 0$, so that

$$S_{\text{inner}} = -\left(\frac{E_0 N I_0}{2\pi a}\right) 2\pi a h = -E_0 N I_0 h$$

Therefore, flux leaving the coil at the outer radius is equal to the flux entering coil at the inner radius, and we have

$$\boxed{\oint_S \mathbf{S}\cdot d\mathbf{a} = 0}$$

7) (20 points) An optical system consists of

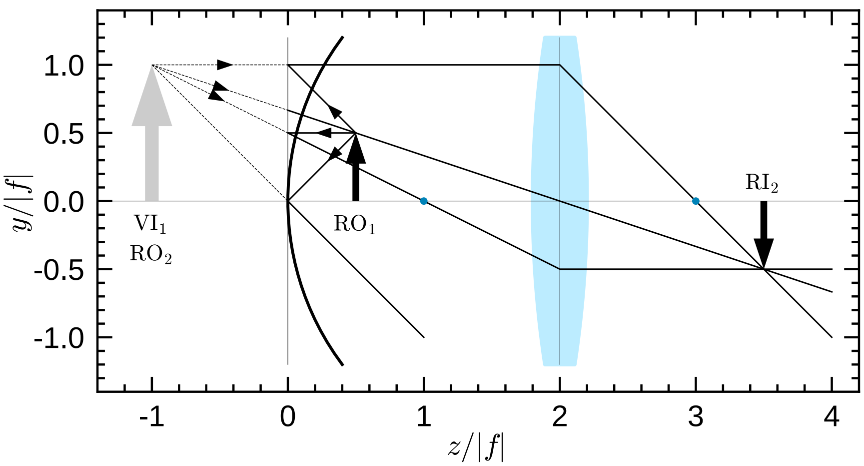
- A converging lens of focal length $f > 0$, located at $z = 2f$.
- A concave mirror, with radius of curvature $R = 2f$ and vertex located at $z = 0$. It is oriented to the *right*, so that its center-of-curvature is at $z = 2f$.
- An upright arrow of height y_1, located in the object plane at $z = 0.5f$.

The arrow only emits light moving toward the mirror, so don't consider any light that initially moves toward the lens.
a) Calculate the location z_2' and total magnification m of the resulting image.
b) Sketch the system, including three principal rays for both the mirror and the lens. The mirror's third principal ray reflects off its vertex. Identify all objects and images as real or virtual using our usual notation (e.g., RO_1 for the real object of subsystem 1). Remember that something can be both an image and an object.
c) A ray (y_1, θ_1) passes through the optical system, beginning in the object plane. Find the ray (y_2, θ_2) in the image plane.

Solution.
a) The final image location is $\boxed{z_2' = 3.5f}$ and the total magnification is $\boxed{m = -1}$.

Subsystem	Focal length	Object location	Image location	Magnification
Concave mirror	$f_1 = f$	$s_1 = f/2$	$s_1' = -f$	$m_1 = 2$
Converging lens	$f_2 = f$	$s_2 = 3f$	$s_2' = 3f/2$	$m_2 = -1/2$

b) The system is sketched below. The three principal rays of both the mirror and the lens are indicated with arrows. In the paraxial approximation, we extend the rays to the mirror's tangent plane.



c) The system's net optical ray transfer matrix that traces rays from the mirror's object plane to the lens's image plane is

$$\begin{aligned} M &= M_2 \cdot M_1 \\ &= \begin{pmatrix} m_2 & 0 \\ -1/f_2 & 1/m_2 \end{pmatrix} \begin{pmatrix} m_1 & 0 \\ -1/f_1 & 1/m_1 \end{pmatrix} \\ &= \begin{pmatrix} -1/2 & 0 \\ -1/f & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1/f & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Then,

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = M \cdot \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} = \boxed{\begin{pmatrix} -y_1 \\ -\theta_1 \end{pmatrix}}$$