• Wait until instructed to begin.
• This exam is closed-book, with no external notes, no external scratch paper, and no electronic devices.
• Use this coversheet for scratch work. If needed, extra scratch paper is available.
• This exam will be curved aggressively so that there will be roughly 30% As, 30% Bs, and 30% Cs. Grades will only be curved upward.
• The TAs and I will provide any requested mathematical identity.
• Paperclip your pages together, in order, including this coversheet on top.
Maxwell’s Equations

\[ \oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \quad \text{Gauss’s law} \]
\[ \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} \quad \text{Faraday’s law} \]
\[ \oint_S \mathbf{B} \cdot d\mathbf{a} = 0 \quad \text{Gauss’s law for} \ \mathbf{B} \]
\[ \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \quad \text{Ampère’s law} \]

Gauss’s law
\[ \theta_i = \theta_r \quad \text{Specular reflection} \]
\[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad \text{Snell’s law} \]
\[ f = \frac{r}{2} \quad \text{Spherical mirror} \]
\[ \frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \quad \text{Thin lens equation} \]
\[ m = -\frac{s'}{s} \quad \text{Lateral magnification} \]
\[ \frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r} \quad \text{Refractive surface} \]
\[ \frac{1}{f} = (n - 1) \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \quad \text{Lensmaker’s equation} \]
\[ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \quad \text{Free propagation} \]
\[ \begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix} \quad \text{Dielectric interface} \]
\[ \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \quad \text{Thin lens} \]
\[ \begin{pmatrix} m & 0 \\ -1/f & 1/m \end{pmatrix} \quad \text{Thin lens system} \]

Electrostatics

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \text{Force on charge} \]
\[ \mathbf{E} = \frac{q\mathbf{R}}{4\pi \varepsilon_0 R^2} \quad \text{Coulomb’s law} \]
\[ d\mathbf{F} = I d\mathbf{l} \times \mathbf{B} \quad \text{Force on current} \]
\[ d\mathbf{B} = \frac{\mu_0 I' d\mathbf{r}' \times \mathbf{R}}{4\pi R^2} \quad \text{Biot-Savart law} \]

Electromagnetic waves

\[ I_d = \varepsilon_0 \frac{d\Phi_E}{dt} \quad \text{Displacement current} \]
\[ c = 1/\sqrt{\mu_0 \varepsilon_0} \quad \text{Speed of light} \]
\[ \mathbf{S} = (1/\mu_0) \mathbf{E} \times \mathbf{B} \quad \text{Poynting vector} \]
\[ u_E = (1/2) \varepsilon_0 E^2 \quad \text{Electric energy density} \]
\[ u_B = B^2/(2\mu_0) \quad \text{Magnetic energy density} \]
\[ v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{c}{n} \quad \text{Speed of light in matter} \]

Sinusoidal EM waves

\[ \mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \]
\[ \mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \]
\[ \mathbf{E}_0 = c \mathbf{B}_0 \times \mathbf{k} \]
\[ \omega = 2\pi f, \ k = 2\pi/\lambda, \ \omega = ck, \ \lambda f = c \]
\[ \langle u \rangle = (1/2) \varepsilon_0 E_0^2, \quad I \equiv \langle S \rangle = \sqrt{\varepsilon_0/\mu_0} E_0^2/2 \]
\[ \langle p_{\text{rad}} \rangle = I/c, \quad \langle p_{\text{rad}}^{\text{rel}} \rangle = 2I/c \]

Interference

\[ \text{OPL} = \int_C n(s) \, ds \quad \text{Optical path length} \]
\[ \lambda_0 = n\lambda \quad \text{Vacuum wavelength} \]
\[ I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \phi \quad \text{Two-beam intensity} \]
\[ I = I_0 \cos^2(\phi/2) \quad \text{Two-beam intensity} \]
\[ \phi(\mathbf{r}) = (2\pi/\lambda_0) \text{OPD} + \Delta \phi \quad \text{Two-beam phase} \]
\[ nd \sin \theta = m\lambda_0 \quad \text{Two-slit constructive} \]
\[ nd \sin \theta = (m + 1/2)\lambda_0 \quad \text{Two-slit destructive} \]

Math

\[ (1 + x)^\alpha = 1 + \alpha x \quad (|x| \ll 1) \]
\[ \langle A(t)B(t) \rangle = (1/2) \text{Re}(\overline{A}B^*) \]
\[ \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \]
\[ \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \]
1) (10 points) A ray of light is incident normally on face $AB$ of a glass prism of refractive index $n$, immersed in air. Find the largest value of the angle $\alpha$ so that the ray is totally internally reflected at face $AC$.

![Diagram of a ray of light incident normally on a glass prism](image)

2) (10 points) An object, possibly either virtual or real, is located at $z = 0$. Find the focal length $f$ and location $z_L$ of a lens such that the resulting image, possibly either virtual or real, is located at $z' = d$ with a magnification of $m = +2$.

![Diagram of a lens and an object](image)

3) (10 points) Two coherent sources $S_1$ and $S_2$ of equal power emit at the same wavelength $\lambda_0$, but they are $180^\circ$ out of phase with each other. Assuming $\lambda_0 \ll d$, find the largest finite value $x > 0$ on the $x$-axis for which there is complete destructive interference. Justify your approximation.

![Diagram of two sources emitting waves](image)
4) (15 points) Light of wavelength $\lambda_0$ is incident in air, from above, on a thin film of thickness $t = 2\lambda_0/3$ and width $w$ placed on a substrate of refractive index $n_0 = 1.6$. If the film’s index of refraction varies over its width according to $n(x) = 1 + (x/w)$, find the two values of $x$ for which the reflected light constructively interferes with the incident light.

![Diagram of light incident on a thin film with varying refractive index.]

5) (15 points) A current $I_0$ flows along the $z$ axis, in the positive $z$ direction, terminating at the origin, where a positive charge $Q(t) = I_0 t$ accumulates. Loop $C$ is a counter-clockwise circular path of radius $a$, centered on the $z$ axis, in the plane $z = d$, as shown. Assuming $a \ll d$, find the line integral of $B$ around loop $C$.

![Diagram of a current $I_0$ flowing along the $z$ axis with a charge $Q$ accumulating at the origin and a loop $C$ in the plane $z = d$.]
6) (20 points) A cutaway view of an ideal toroidal coil, whose axis-of-symmetry coincides with the $z$ axis, is shown below. It has rectangular cross-section, inner radius $a$, outer radius $b$, height $c$, current $I_0$, and a total of $N$ turns. There is an electric field $E = -E_0\hat{z}$.

a) Find the Poynting vector $S$ everywhere.

b) Calculate the flux of the Poynting vector through the toroid’s inner surface, oriented in the outward direction,

$$\oint_S S \cdot da$$
7) (20 points) An optical system consists of

- A converging lens of focal length $f > 0$, located at $z = 2f$.
- A concave mirror, with radius of curvature $R = 2f$ and vertex located at $z = 0$. It is oriented to the right, so that its center-of-curvature is at $z = 2f$.
- An upright arrow of height $y_1$, located in the object plane at $z = 0.5f$.

The arrow only emits light moving toward the mirror, so don’t consider any light that initially moves toward the lens.

a) Calculate the location $z'_2$ and total magnification $m$ of the resulting image.

b) Sketch the system, including three principal rays for both the mirror and the lens. The mirror’s third principal ray reflects off its vertex. Identify all objects and images as real or virtual using our usual notation (e.g., RO$_1$ for the real object of subsystem 1). Remember that something can be both image and an object.

c) A ray $(y_1, \theta_1)$ passes through the optical system, beginning in the object plane. Find the ray $(y_2, \theta_2)$ in the image plane.