Physics 1C - First Midterm

Thursday, April 27, 10-11:50AM

UCLA / Spring 2023 / Brian Naranjo

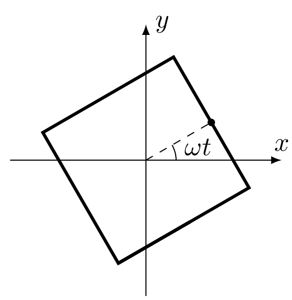
Solutions[1]

1) (10 points) A charge q is moving with velocity $\mathbf{v} = v_0 \cos\alpha\,\hat{\mathbf{x}} + v_0 \sin\alpha\,\hat{\mathbf{z}}$ in uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{x}}$. If we want the particle to maintain the same velocity, what electric field should we introduce?

Solution. To maintain the same velocity, we should introduce an electric field such that net force on the particle is zero,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0 \implies \mathbf{E} = -\mathbf{v} \times \mathbf{B} = -v_0 B \sin\alpha(\hat{\mathbf{z}} \times \hat{\mathbf{x}}) = \boxed{-v_0 B \sin\alpha\,\hat{\mathbf{y}}}$$

2) (10 points) A square loop of side length a and resistance R is centered on the origin and is rotating counterclockwise in the xy plane with angular velocity ω, as shown. If there is a magnetic field $\mathbf{B} = B_0 \cos\omega t\,\hat{\mathbf{z}}$, find the magnitude *and* direction of the loop's induced current.



Solution. The loop's rotation is irrelevant in applying Faraday's flux rule, as the net flux through the loop does not depend on the loop's angle-of-rotation. The net emf is due to the electric field induced by the time-varying magnetic field. This, however, does not mean that the motional emf is zero throughout the loop. There are portions of the loop having a nonzero motional emf, but the net motional emf along each side sums to zero.

Taking the loop orientation to be counterclockwise, the magnetic flux through the loop is

$$\Phi(t) = \int_S \mathbf{B}(t) \cdot d\mathbf{a} = B_0 a^2 \cos\omega t.$$

By Faraday's flux rule, the loop's emf is

$$\mathcal{E} = -\frac{d\Phi}{dt} = B_0 a^2 \omega \sin\omega t.$$

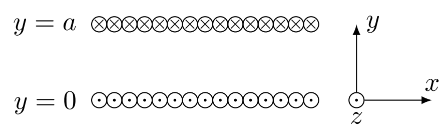
Using $\mathcal{E} = IR$, the *positive* sense of current in the loop is $\boxed{\text{counterclockwise}}$ and has magnitude

$$\boxed{I(t) = \frac{B_0 a^2 \omega}{R} \sin\omega t}$$

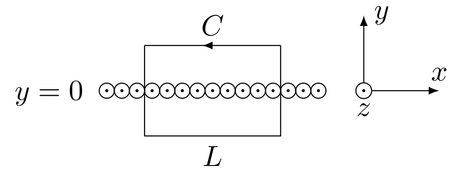
Using Lenz's law, we confirm the direction of the induced current. Just after $t = 0$, the flux is *decreasing* in the $\hat{\mathbf{z}}$ direction, so that current is induced in the counterclockwise direction to restore some of this lost flux, as predicted by Faraday's flux rule.

[1] Though this pedagogical write-up is lengthy, these problems don't involve lengthy calculations. Correct solutions on the submitted exams are typically quite short.

3) (10 points) Consider *two* infinite sheets of tightly packed wires. In the lower sheet, located in the plane $y = 0$, each wire is parallel to the z axis and carries current I in the positive z direction. In the upper sheet, located in the plane $y = a$, each wire is also parallel to z axis but instead carries current $2I$ in the *negative* z direction. Along the x direction, each sheet separately carries n wires per unit length. Find the magnetic field $\mathbf{B}$ everywhere.

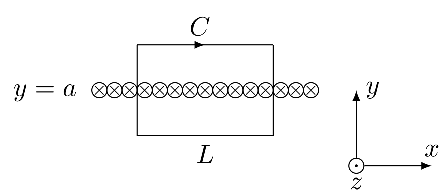


Solution. Our strategy is to find, by Ampère's law, the magnetic field about each sheet current considered separately, and then combine the two fields, using the *principle of superposition.*



Using the Amperian loop shown above, the magnetic field due to the lower sheet is

$$\mathbf{B}_{\text{upper}}(y) = \begin{cases} -\dfrac{\mu_0 n I}{2}\hat{\mathbf{x}} & y > 0 \\ \dfrac{\mu_0 n I}{2}\hat{\mathbf{x}} & y < 0 \end{cases}$$



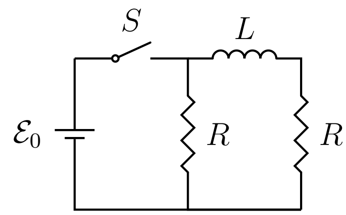
Using the Amperian loop shown above, the magnetic field due to the upper sheet is

$$\mathbf{B}_{\text{lower}}(y) = \begin{cases} \mu_0 n I\hat{\mathbf{x}} & y > a \\ -\mu_0 n I\hat{\mathbf{x}} & y < a \end{cases}$$

Summing the two fields,

$$\boxed{\mathbf{B}(y) = \begin{cases} \dfrac{\mu_0 n I}{2}\hat{\mathbf{x}} & y > a \\ -\dfrac{3\mu_0 n I}{2}\hat{\mathbf{x}} & 0 < y < a \\ -\dfrac{\mu_0 n I}{2}\hat{\mathbf{x}} & y < 0 \end{cases}}$$

4) (15 points) In the circuit below, switch S is closed shut for a very long time, so that the currents are steady. Then, at $t = 0$, the switch is abruptly opened. Find, by any method, the subsequent current $I_L(t)$ through the inductor, and indicate whether the current is flowing clockwise or counterclockwise.



Solution. For LRC transient problems in Physics 1C, we adopt a holistic method of solution. Just before $t = 0$, the currents are steady, so that we can replace the inductor with a dead short. The circuit's effective resistance is obtained by adding the two resistors in parallel. We find $R_{\text{eff}} = R/2$. The current through the battery is then $\mathcal{E}_0/R_{\text{eff}} = 2\mathcal{E}_0/R$ and the current through the inductor half of that.

An inductor's induced emf acts to oppose changes in the current passing through the inductor. A discontinuous change in current would be opposed with an infinite induced emf. Therefore, the value of the inductor's current is continuous across the closing of the switch, and we have

$$I_L(t = 0) = I_0 = \mathcal{E}_0/R,$$

flowing in the clockwise direction.

Once the switch is closed, we recognize that we have an LR circuit. As time goes on, energy is dissipated in the two resistors and the current decays exponentially with time constant $\tau_L = L/(2R)$, with the current reaching zero asymptotically as $t \to \infty$.

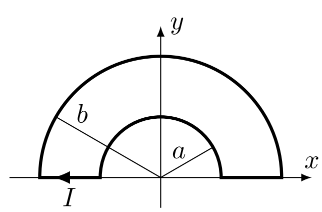
Combining these observations, current flows in the $\boxed{\text{clockwise}}$ direction, with

$$\boxed{I_L(t > 0) = I_0 e^{-t/\tau_L}},$$

where

$$\boxed{I_0 = \mathcal{E}_0/R} \text{ and } \boxed{\tau_L = L/(2R)}.$$

5) (15 points) A current I flows clockwise in the loop shown below. Using the Biot-Savart law, write a derivation for the magnetic field $\mathbf{B}$ at the origin.



Solution. The Biot-Savart law is

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I' d\mathbf{r}' \times \hat{\mathbf{R}}}{R^2}$$

The straight portions do not contribute to magnetic field at the origin because, along both of these segments, $d\mathbf{r}'$ and $\mathbf{R}$ are parallel, and therefore, their cross product is zero.

Let's first evaluate the magnetic field due to the inner semicircle. The relative displacement vector, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, has magnitude $R = a$ and points from the source point, along the integration path, back to the observation point at $\mathbf{r} = 0$. The infinitesimal displacement along the path, $d\mathbf{r}' = a\, d\phi\, \hat{\boldsymbol{\phi}}$, is perpendicular to $\hat{\mathbf{R}}$. Their cross-product is in the $\hat{\mathbf{z}}$ direction. Combining these,

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\, d\phi}{a} \hat{\mathbf{z}}$$

In lecture (W01 and RP1), we calculated Biot-Savart integrands using a more formal step-by-step procedure, identifying all vectors and carefully evaluating all magnitudes and cross products. Here, in this relatively simple case, we have more relied on intuition and geometric reasoning, which comes with practice. You are welcome to use either method.
Then, the magnetic field due to the inner semicircle is

$$\mathbf{B}_{\text{inner}} = \frac{\mu_0}{4\pi}\frac{I}{a}\hat{\mathbf{z}}\int_0^{\pi} d\phi = \frac{\mu_0 I}{4a}\hat{\mathbf{z}}$$

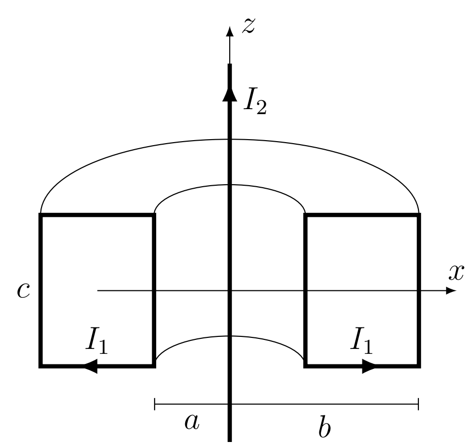
Continuing clockwise, the magnetic field due to the outer semicircle is taken in the reverse direction, from $\phi = \pi$ to $\phi = 0$, so

$$\mathbf{B}_{\text{outer}} = -\frac{\mu_0 I}{4b}\hat{\mathbf{z}}$$

Combining the two fields,

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{\mathbf{z}} = \frac{\mu_0 I}{4}\left(\frac{b-a}{ab}\right)\hat{\mathbf{z}}}$$

6) (20 points) A cutaway view of an ideal toroidal coil, whose axis-of-symmetry coincides with the z axis, is shown below. It has rectangular cross-section, inner radius a, outer radius b, height c, and a total of N turns. In addition to the toroidal coil, there is a long straight wire that coincides with the z axis. The directions of *positive* currents I_1 and I_2 in these conductors are indicated with arrows
a) What is the mutual inductance M of the toroidal coil and the long straight wire?
b) If we send a pulse of current through the toroidal coil, $I_1(t) = I_0 \exp[-t^2/(2\tau^2)]$, then, assuming that the long straight wire has total resistance R_2, find the current $I_2(t)$ induced in the straight wire.



Solution.
a) Assume a current I_2 flows in the straight wire. Its magnetic field is

$$\mathbf{B} = \frac{\mu_0 I_2}{2\pi r}\hat{\boldsymbol{\phi}},$$

and its flux through the toroidal coil is

$$\Phi_{2\to 1} = N\int_S \mathbf{B}\cdot d\mathbf{a} = -\frac{\mu_0 N I_2}{2\pi}c\int_a^b \frac{dr}{r} = -\frac{\mu_0 N c\ln(b/a)}{2\pi} I_2 = M_{12} I_2$$

By reciprocity, $M = M_{12} = M_{21}$. Therefore, we identify

$$\boxed{M = -\frac{\mu_0 N c \ln(b/a)}{2\pi}}$$

Note that the negative sign originates with the given loop orientations. In this case, the azimuthal magnetic field due to current I_2 is pointed in the *opposite* direction of the toroid's integration surface S, obtained from the Stokes' theorem's right-hand rule for the toroid's loop orientation. If we flip one of the loop's orientations, then M would become positive. This is in contrast to any loop's self-induction L, which is *always* positive.

As usual, we could have *also* calculated the mutual inductance M by instead assuming a current I_1 flowing in the toroidal coil and then calculating the resulting flux through circuit C_2, which is formed by the long straight wire and a big semicircular half-loop at infinity. In this case, the work is the same, but, perhaps, it is a bit more conceptually sketchy.

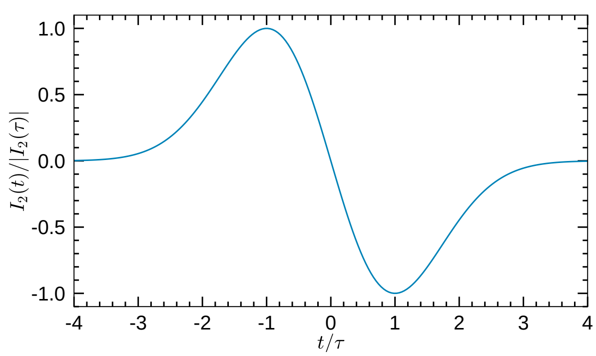
b) The induced emf in the straight wire is

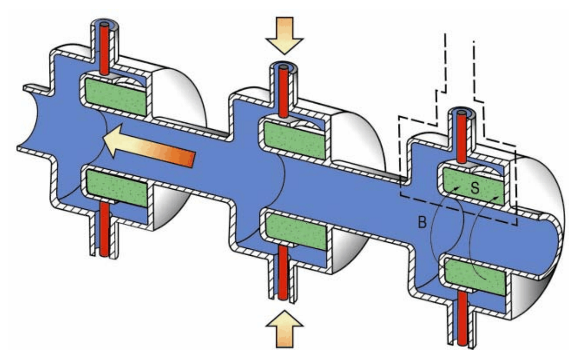
$$\mathcal{E}_{1\to 2} = -\frac{d\Phi_{1\to 2}}{dt} = -\frac{d(M_{21} I_1)}{dt} = -M\frac{dI_1}{dt} = I_2(t) R_2$$

Therefore,

$$\boxed{I_2(t) = -\frac{M}{R_2}\frac{dI_1}{dt} = \left(\frac{I_0 M}{R_2 \tau^2}\right) t e^{-t^2/(2\tau^2)} = -\left(\frac{I_0 \mu_0 N c \ln(b/a)}{2\pi R_2 \tau^2}\right) t e^{-t^2/(2\tau^2)}}$$

The current induced in the straight wire is shown on the left below.



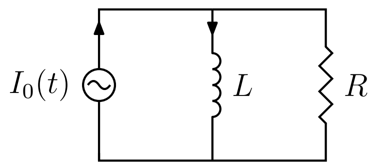


As the toroid's current is *increasing*, for $t < 0$, current is induced on the straight wire in the $\hat{\mathbf{z}}$ direction to oppose this change. After the pulse's peak, the toroid's current is *decreasing*, and current is induced on the straight wire in the $-\hat{\mathbf{z}}$ direction. The induced current's peak magnitude occurs at $t = \pm\tau$.

Shown above, on the right, is a type of particle accelerator called an **induction linac**, first built at Lawrence Livermore National Laboratory. Synchronized current pulses are sent through a sequence of toroidal coils, which, in turn, induce an intense electric field along the axis, used to accelerate charged particles.

7) (20 points) Consider a current source $I_0(t) = I_0 \cos\omega t$ driving the circuit below, where the positive sense of current is indicated with arrows. Find the real-valued amplitude I_L and phase ϕ_L such that the current through the inductor is written

$$I_L(t) = I_L \cos(\omega t - \phi_L).$$



Solution. Represent the current source with phasor

$$\widetilde{I}_0 = I_0 e^{i\omega t}.$$

The circuit's total impedance is found by adding the inductor's and resistor's impedances in parallel,

$$\frac{1}{Z_0} = \frac{1}{Z_L} + \frac{1}{Z_R}$$

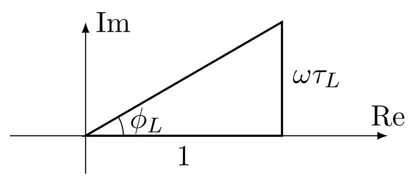
The source's current and voltage are related through Ohm's law,

$$\widetilde{\mathcal{E}}_0 = \widetilde{I}_0 Z_0.$$

Then, the current through the inductor is also found from Ohm's law,

$$\widetilde{I}_L = \frac{\widetilde{\mathcal{E}}_0}{Z_L} = \frac{Z_0}{Z_L}\widetilde{I}_0 = \frac{\widetilde{I}_0}{1 + (Z_L/Z_R)} = \frac{I_0 e^{i\omega t}}{1 + i\omega\tau_L}$$

To express $\widetilde{I}_L$ in polar form, we first express the complex denominator in polar form. Using the diagram,



So,

$$1 + i\omega\tau_L = \sqrt{1 + (\omega\tau_L)^2}\; e^{i\phi_L},$$

where

$$\boxed{\phi_L = \arctan(\omega\tau_L)}.$$

Now, we have $\widetilde{I}_L$ in polar form,

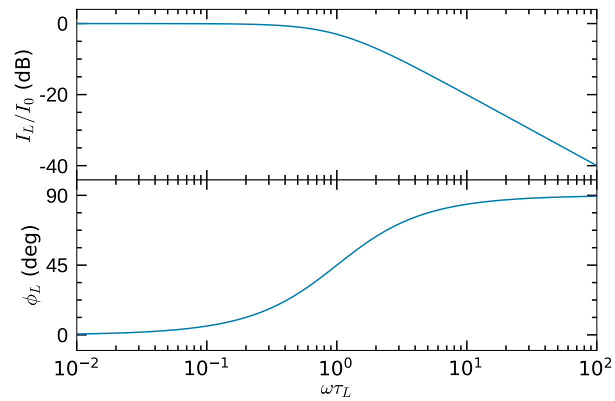
$$\widetilde{I}_L = \frac{I_0}{\sqrt{1 + (\omega\tau_L)^2}} e^{i(\omega t - \phi_L)},$$

and $I_L(t)$ is determined by taking the real component

$$I_L(t) = \mathrm{Re}\,\widetilde{I}_L = I_L \cos(\omega t - \phi_L)$$

where

$$\boxed{I_L = \frac{I_0}{\sqrt{1 + (\omega\tau_L)^2}}}$$



Shown above is the circuit's frequency response. At very low frequencies, $\omega\tau_L \ll 1$, the inductor acts as a dead short, and the current through the inductor is the same as the source's current. At higher frequencies, it begins to get harder to drive current through the inductor. We say that the inductor's current *rolls off* at frequency $\omega = 1/\tau_L$.