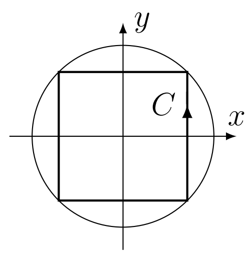
# Physics 1C - Final

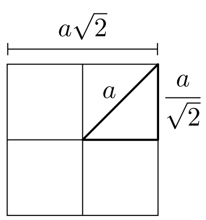
Tuesday, June 13, 11:30AM-2:30PM

UCLA / Spring 2023 / Brian Naranjo

## Solutions

**1)** (8 points) Consider a thick cylindrical wire of radius $a$ whose axis coincides with the $z$ axis. Uniformly over its cross section, it carries a current $I_0$ in the positive $z$ direction, generating an azimuthal magnetic field $\mathbf{B}$. In the $xy$ plane, path $C$ is a square inscribed in the wire's cross section, oriented as shown. Find the line integral of the magnetic field around path $C$.

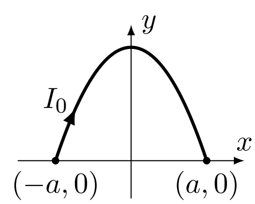



**Solution.** The square's side length is $a\sqrt{2}$. The current through the square is proportional to the ratio of the square's area, $2a^2$, to the circle's area, $\pi a^2$. Then, Ampère's law gives

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} = \frac{\mu_0 I_0 A_{\text{square}}}{A_{\text{circle}}} = \boxed{\frac{2\mu_0 I_0}{\pi}}$$

**2)** (8 points) A wire in the $xy$ plane carries current $I_0$ from its initial point at $(-a, 0)$ to its final point at $(a, 0)$. The wire is bent into a parabola satisfying $y = y_0\,[1 - (x/a)^2]$, as shown below. Find the net force on the wire if it is immersed in a magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$.



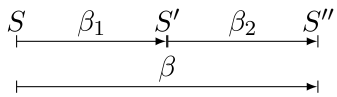

**Solution.** No need to go sweaty mode here. Like we saw in PS1 Q3, the uniform magnetic field comes out of the integral, which then trivially becomes the path's overall displacement.

$$\mathbf{F}_{\text{net}} = \int_C I\, d\mathbf{r} \times \mathbf{B} = I_0 \left( \int_C d\mathbf{r} \right) \times \mathbf{B} = I_0(2a\hat{\mathbf{x}}) \times (B_0\hat{\mathbf{z}}) = \boxed{-2I_0B_0a\hat{\mathbf{y}}}$$

**3)** (8 points) Alice and Bob rocket toward each other at a very high speed. From Alice's frame, Bob approaches at a speed equal to $(4/5)c$. Likewise, from Bob's frame, Alice approaches at a speed equal to $(4/5)c$. Meanwhile, you are stationed between Alice and Bob, where you note that Alice and Bob are both approaching you at the *same* speed. In terms of $c$, what is that speed?

**Solution.** This is essentially the same as MP8 Q9 (Y&F 37.21), whose solution was posted on Campuswire. In this problem, we take

$$S = \text{Alice}, \; S' = \text{You}, \; \text{and } S'' = \text{Bob}.$$




The given information says $\beta = -4/5$ and $\beta_1 = \beta_2 < 0$. Then, the velocity addition rule gives

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \frac{2\beta_1}{1 + \beta_1^2} \implies \beta\beta_1^2 - 2\beta_1 + \beta = 0$$

Using the quadratic formula,

$$\beta_1 = \frac{2 \pm \sqrt{4 - 4\beta^2}}{2\beta} = \frac{1}{\beta} \pm \sqrt{\left(\frac{1}{\beta}\right)^2 - 1} = -\frac{5}{4} \pm \frac{3}{4} = -\frac{1}{2},$$

where we have excluded the unphysical solution $\beta_1 = -2$. In your frame, the speed of either Alice or Bob is then $\boxed{v_1 = |\beta_1|c = c/2}$

**4)** (10 points) A double slit is uniformly and coherently illuminated by light of wavelength $\lambda_0$. The slits are spaced a distance $d = 15\lambda_0$ apart and the width of each slit is $a = 3\lambda_0$. How many interference maxima lie within the central diffraction envelope?

**Solution.** This is essentially the same as an example presented during W09 lecture. The observed intensity pattern is

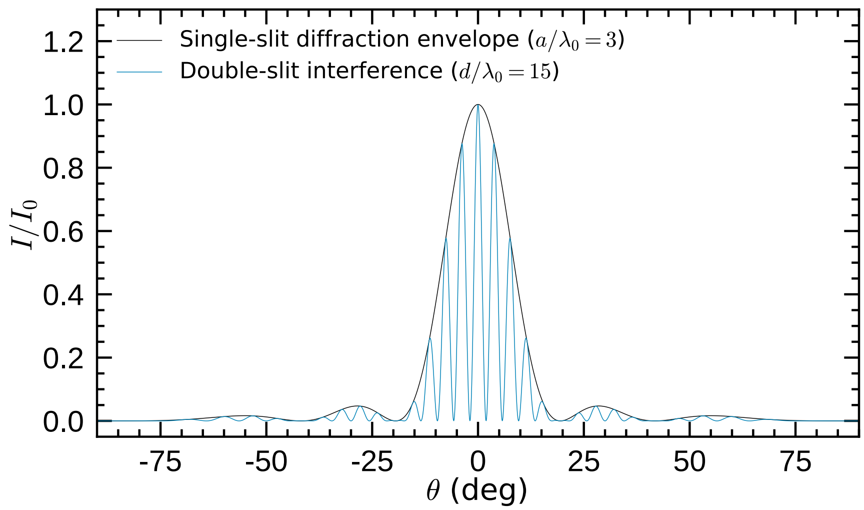
$$I = I_0 \cos^2(\phi/2)\,\mathrm{sinc}^2(\beta/2).$$

The *interference* peaks are governed by the phase difference between the two slits,

$$\phi = \left(\frac{2\pi}{\lambda_0}\right) d \sin\theta,$$

whereas the *diffraction* envelope is governed by the phase difference between opposite ends of each slit,

$$\beta = \left(\frac{2\pi}{\lambda_0}\right) a \sin\theta.$$




Maxima of $\cos^2 x$ occur at $x = 0, \pm\pi, \pm 2\pi, \ldots$ Therefore, interference peaks are located at

$$m\pi = \frac{\phi}{2} \implies \sin\theta_m = \frac{m\lambda_0}{d} = \frac{m}{15}, \text{ where } m = 0, \pm 1, \pm 2, \ldots$$

The first zero of $\operatorname{sinc} x$ occurs at $x = \pm\pi$. Therefore, central diffraction envelope extends from $-\theta_a$ to $\theta_a$, where

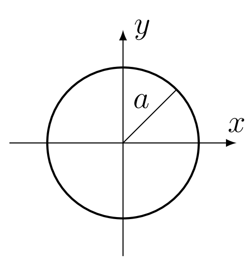
$$\pi = \frac{\beta}{2} \implies \sin\theta_a = \frac{\lambda_0}{a} = \frac{1}{3}$$

The interference peaks at $\theta_{\pm 5}$ coincide with the diffraction minima, so we don't include them in our sum. The interference peaks within the central diffraction envelope then occur at

$$\sin\theta_m = \left\{ -\frac{4}{15}, -\frac{3}{15}, -\frac{2}{15}, \ldots, \frac{4}{15} \right\},$$

for a total of $\boxed{4 + 1 + 4 = 9}$.

**5)** (10 points) Consider a volume $V$ of a metal with electrical conductivity $\sigma$. We draw the metal out into a uniformly thin wire formed into a circular loop of radius $a$ in the $xy$ plane, as shown. If the wire is then immersed in a time-varying magnetic field $\mathbf{B}(t) = \hat{\mathbf{z}} B_0 t/\tau$, find the magnitude and the direction of the induced current flowing in the circular loop.




**Solution.** This is an easier variation on PS2 Q2. Because the wire is thin, to a good approximation, then $V = A\, 2\pi a$, where $A$ is the wire's cross-section area. The Ohmic resistance of the circular loop is is then

$$R = \frac{L}{\sigma A} = \frac{2\pi a}{\sigma A} = \frac{4\pi^2 a^2}{\sigma V}.$$

Taking the orientation of the loop to be counterclockwise, the magnetic flux through the loop is

$$\Phi = \frac{B_0 t}{\tau} \pi a^2$$

Then, Faraday's flux rule gives the net emf around the loop,

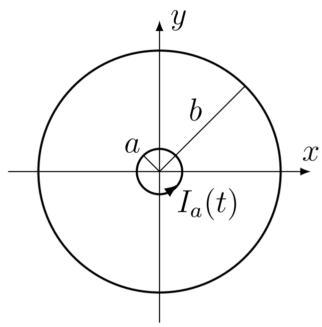
$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{B_0}{\tau}\pi a^2 = IR$$

Because of the negative sign, the current is $\boxed{\text{clockwise}}$, opposite of our assumed orientation. We confirm this using Lenz's law. The induced current should generate a flux *into* the page, in opposition to the steadily growing flux *out* of the page. Indeed, a clockwise current generates a magnetic field into the page. The magnitude of the current is

$$|I| = \frac{B_0\pi a^2}{\tau R} = \boxed{\frac{B_0\sigma V}{\tau 4\pi}}$$

It is somewhat amusing that the current is independent of the loop's radius. A larger loop collects more flux, with the resulting emf proportional to $a^2$. On the other hand, a larger loop is made with a longer, thinner wire, which both increase the overall resistance, proportional to $a^2$. Both effects exactly cancel.

**6)** (12 points) Consider two circular conducting loops in the $xy$ plane, centered on the origin. The inner loop has radius $a$, and the outer loop has radius $b$. You may assume that $a \ll b$. A time varying current $I_a(t) = I_0 \cos\omega t$ flows through the inner loop, where we take the positive sense of current to be in the counterclockwise direction. If $R$ is the resistance of the outer loop, then find the magnitude and direction of current $I_b(t)$ induced in the outer loop.




**Solution.** This is conceptually the same as PS2 Q4. Let's first choose the orientation of the outer loop to also be counterclockwise. Then, given an inner loop current $I_a$, the resulting flux through the outer loop is

$$\Phi_{a\to b} = M_{ba} I_a.$$

Faraday's flux rule yields the current through the outer loop,

$$\mathcal{E}_{a\to b} = -\frac{d\Phi_{a\to b}}{dt} = -M_{ba}\frac{dI_a}{dt} = I_b R \implies I_b = -\frac{M_{ba}}{R}\frac{dI_a}{dt}.$$

Now, we only need to calculate $M_{ba}$. Given an inner loop current $I_a$, however, directly finding the resulting flux through the outer loop seems extraordinarily difficult and may even not be properly defined. Inside the inner loop, the magnetic field comes out of the page. Outside the inner loop, the magnetic field goes into the page. The first difficulty we encounter is that the magnetic field diverges at the inner loop. We could perhaps overcome this difficulty by calculating the flux by initially assuming that the inner wire has a finite thickness, and then taking the limit for a thin inner loop wire. The second difficulty is that we still need to find the field strength as a function of $x$ in a region where we won't be able to take advantage of $a \ll b$.

Fortunately, we may instead use the *reciprocity* property of mutual inductance,

$$M_{ba} = M_{ab}.$$

As we shall see shortly, finding $M_{ab}$ is relatively painless. It is defined by relating an assumed current $I_b$ through the outer loop to the resulting flux through the inner loop,

$$\Phi_{b\to a} = M_{ab} I_b$$

We assume that the inner loop is small enough that, to a good approximation, the resulting magnetic field is constant over its area. By the Biot-Savart law, the direction of the magnetic field is in the $\hat{\mathbf{z}}$ direction, and its magnitude is found by integrating around the outer loop,

$$dB = \frac{\mu_0}{4\pi}\frac{I_b dr'}{b^2} \implies B = \frac{\mu_0 I_b}{2b}$$

Then,

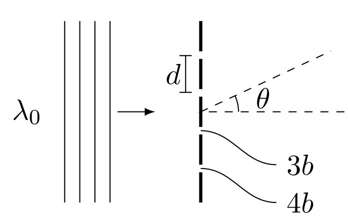
$$\Phi_{b\to a} = \left(\frac{\mu_0 I_b}{2b}\right)\pi a^2 \implies M_{ab} = \frac{\mu_0 \pi a^2}{2b}$$

Finally,

$$I_b = -\frac{M_{ab}}{R}\frac{dI_a}{dt} = \boxed{\frac{\mu_0 \pi a^2 I_0 \omega}{2bR}\sin\omega t},$$

where the positive sense of current is in the $\boxed{\text{counterclockwise}}$ direction.

**7)** (12 points) Consider four thin slits that are uniformly spaced a distance $d$ apart. The slits are coherently and uniformly illuminated by light of wavelength $\lambda_0$. We then cover each slit with a thin dielectric plate of refractive index $n$. The first plate's thickness is $b$, the second plate's thickness is $2b$, etc. The resulting intensity is observed at angle $\theta$, as shown below.
**a)** At what angle is the beam's central maximum $\theta_0$?
**b)** What is the angular separation $\Delta\theta_0$ between the central maximum and either one of its adjacent minima?




**Solution.** This is an extension of PS6 Q3 from $N = 3$ to $N = 4$, though, it is slightly complicated by the additional beam-to-beam OPD dependent on the angle of observation. On the other hand, there is little computation here because we can simply quote the W09 general result for $N$ beam interference.

Conceptually, we have $N = 4$ coherent beams of equal intensity and a uniform phase increment $\phi$. In W09, we found the resulting intensity, as a function of $\phi$, by summing the phasors, written as a finite geometric series. As given on the cheat sheet, the intensity is

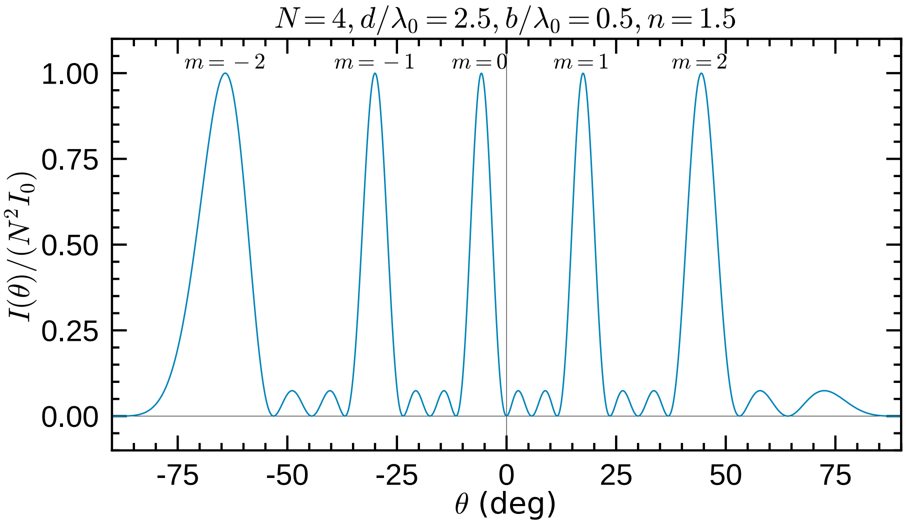
$$I = I_0 \frac{\sin^2(N\phi/2)}{\sin^2(\phi/2)}.$$

In addition to the usual phase increment due to neighboring slits spaced a distance $d$ apart, the phase increment now also includes a term accounting for the OPD resulting from the dielectric plates,

$$\phi = \left(\frac{2\pi}{\lambda_0}\right)\left[d\sin\theta + (n-1)b\right]$$

The quantity in the brackets is the optical path difference between neighboring slits, $\mathrm{OPD} = \mathrm{OPL}_{k+1} - \mathrm{OPL}_k$. According to the diagram above, the optical path length due to both terms increases as we move from the first slit, at the top, to the fourth slit, at the bottom. As we progress downwards, for each subsequent slit, we "convert" a thickness $b$ of vacuum, with dielectric constant 1, to a thickness $b$ of dielectric plate, with dielectric constant $n$. This yields the mysterious factor $n-1$. It is a common mistake to use $n$ instead of $n-1$ here. A good check is to make sure that your result reduces to the usual result if the dielectric plates were made of vacuum, $n = 1$.

As shown below, the presence of the plates shifts the overall interference pattern, functionally equivalent to a *phased-array antenna*. You explored the continuum version of this system in PS7 Q1.



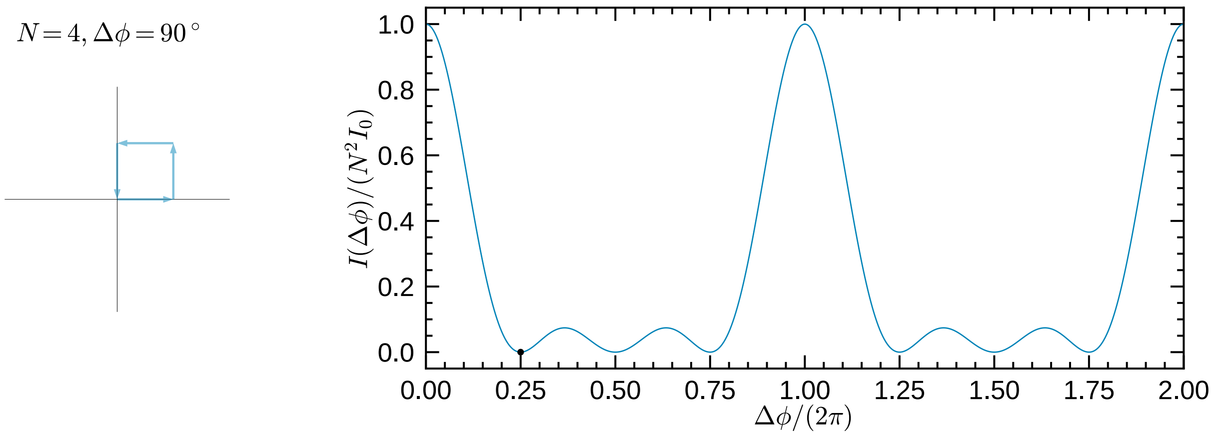

**a)** The central maximum or zeroth order, $m = 0$, occurs when $\phi = 0$,

$$d\sin\theta_0 + (n-1)b = 0 \implies \boxed{\theta_0 = -\arcsin\left[\frac{(n-1)b}{d}\right]}$$

In the example case shown in the above diagram,

$$\theta_0 = -\arcsin\left[\frac{(1.5-1)0.5}{2.5}\right] \approx -5.74^\circ$$

**b)** Generally, apart from the constructive peaks at integer multiples of $\phi = 2\pi$, the intensity $I$ has destructive minima at multiples of $\phi = 2\pi/N$. In our case, $N = 4$, the minima adjacent to the constructive peaks corresponds to $\phi = 90^\circ$, where the four phasors form a square, shown below




Alternatively, the first zero is easily seen from the intensity's functional form,

$$\sin(N\phi/2) = 0 \implies N\phi_\pm/2 = \pm\pi \implies \phi_\pm = \pm 2\pi/N$$

Then, the adjacent minima are located at $\theta_\pm$, determined by

$$\phi_\pm = \pm\frac{2\pi}{N} = \left(\frac{2\pi}{\lambda_0}\right)[d\sin\theta_\pm + (n-1)b] \implies \boxed{\theta_\pm = \arcsin\left[\pm\frac{\lambda_0}{4d} - \frac{(n-1)b}{d}\right]}$$

The upper and lower angular separation, without approximations, are then

$$\boxed{\Delta\theta_\pm = |\theta_\pm - \theta_0|}.$$

You were only asked to find one of these, so it is fine if you only wrote down one of them. I will now develop an approximation for $\Delta\theta \approx \Delta\theta_\pm$. You weren't asked to develop an approximation, so the remaining solution is just for fun.

Let

$$x_0 = -\frac{(n-1)b}{d} \quad \text{and} \quad \Delta x = \frac{\lambda_0}{4d}$$

so that

$$\theta_\pm = \arcsin(x_0 \pm \Delta x) \quad \text{and} \quad \theta_0 = \arcsin(x_0),$$

Then,

$$\Delta\theta_\pm = |\theta_\pm - \theta_0| = |\arcsin(x_0 \pm \Delta x) - \arcsin(x_0)|$$

Using[1]

$$\arcsin(x+y) = \arcsin(x) + \frac{y}{\sqrt{1-x^2}} + \mathcal{O}(y^2),$$

$$\boxed{\Delta\theta_\pm \approx \Delta\theta \equiv \frac{(\lambda_0/4)}{\sqrt{d^2 - (n-1)^2 b^2}}}$$

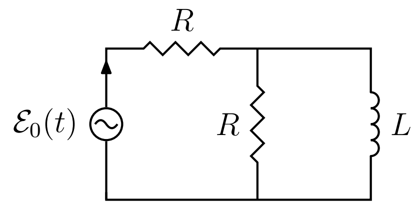
which is a good approximation for $\lambda_0/(4d) \ll 1$. In the example case shown earlier, $\lambda_0/(4d) = 0.1$, and

$$\Delta\theta = \frac{1/4}{\sqrt{2.5^2 - (1.5-1)^2 0.5^2}} \approx 5.8^\circ$$

[1] `Series[ArcSin[x + y], {y, 0, 3}]`

**8)** (16 points) Consider a voltage source $\mathcal{E}_0(t) = \mathcal{E}_0 \cos \omega t$ driving the circuit below, where the positive sense of current is indicated with an arrow. Assuming the inductor's top terminal is its positive reference, find the real-valued amplitude $V_L$ and phase $\phi_L$ such that the voltage across the inductor is written

$$V_L(t) = V_L \cos(\omega t + \phi_L).$$




**Solution.** Represent the voltage source with phasor

$$\widetilde{\mathcal{E}}_0 = \mathcal{E}_0 e^{i\omega t}.$$

The circuit's total impedance $Z_0$ is

$$Z_0 = R + Z_1 \text{ where } \frac{1}{Z_1} = \frac{1}{R} + \frac{1}{Z_L} \text{ and } Z_L = i\omega L.$$

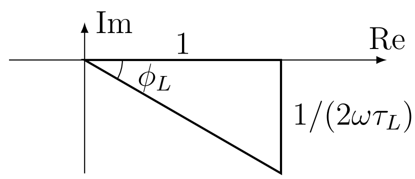
The source's current and voltage are related through Ohm's law,

$$\widetilde{\mathcal{E}}_0 = \widetilde{I}_0 Z_0.$$

Then, the voltage across the inductor is also found from Ohm's law,

$$\widetilde{V}_L = \widetilde{I}_0 Z_1 = \frac{Z_1}{Z_0}\widetilde{\mathcal{E}}_0 = \frac{\widetilde{\mathcal{E}}_0}{1 + (R/Z_1)} = \frac{(\mathcal{E}_0/2)e^{i\omega t}}{1 - i/(2\omega\tau_L)}$$

To express $\widetilde{V}_L$ in polar form, we first express the complex denominator in polar form. Using the diagram,




So,

$$1 - \frac{i}{2\omega\tau_L} = \sqrt{1 + 1/(2\omega\tau_L)^2}\; e^{-i\phi_L},$$

where

$$\boxed{\phi_L = \arctan\left(\frac{1}{2\omega\tau_L}\right)}.$$

Now, we have $\widetilde{V}_L$ in polar form,

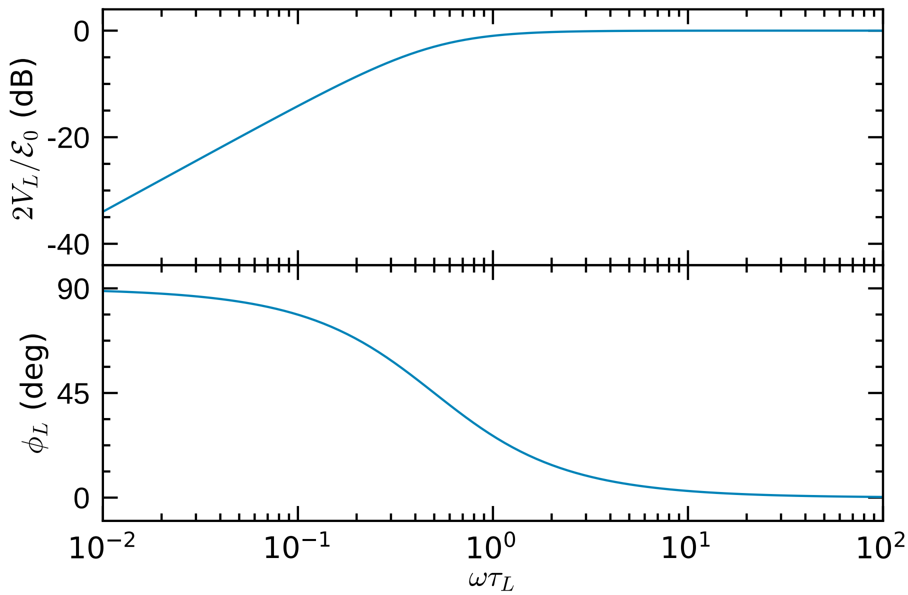
$$\widetilde{V}_L = \frac{\mathcal{E}_0/2}{\sqrt{1 + 1/(2\omega\tau_L)^2}} e^{i(\omega t + \phi_L)},$$

and $V_L(t)$ is determined by taking the real component

$$V_L(t) = \operatorname{Re}\widetilde{V}_L = V_L \cos(\omega t + \phi_L)$$

where

$$\boxed{V_L = \frac{\mathcal{E}_0/2}{\sqrt{1 + 1/(2\omega\tau_L)^2}}}$$




Shown above is the circuit's frequency response. At very low frequencies, $\omega\tau_L \ll 1$, the inductor acts as a dead short, and the voltage across the inductor goes to zero. At higher frequencies, it begins to get harder to drive current through the inductor. Then, the circuit becomes approximately two resistors $R$ in series. As expected, the voltage across the inductor is then equal to $\mathcal{E}_0/2$ and is in-phase with the voltage source.

**9)** (16 points) An optical system consists of

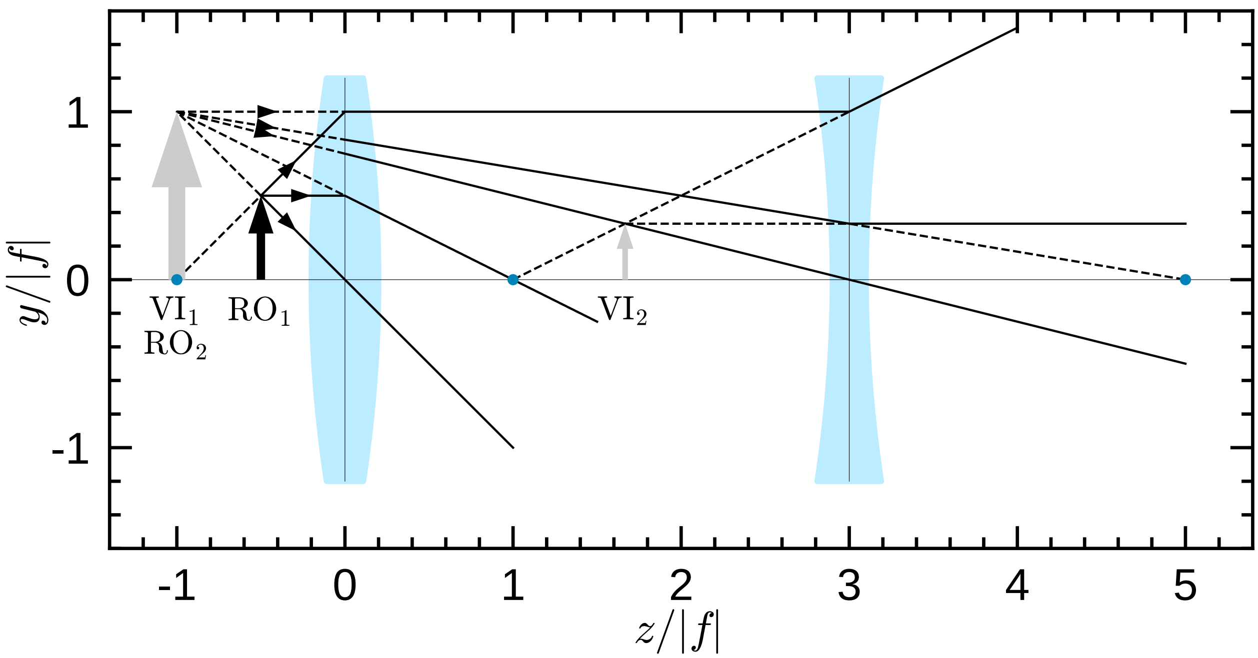
- A converging lens of focal length $f_1 = f > 0$, located at $z = 0$.
- A diverging lens of focal length $f_2 = -2f < 0$, located at $z = 3f$.
- An upright arrow of height $y_1$, located in the object plane at $z_1 = -f/2$.

**a)** Calculate the location $z_2'$ and total magnification $m$ of the resulting image.
**b)** Sketch the system, including three principal rays for each lens. Identify all objects and images as real or virtual using our usual notation (e.g., $\mathrm{RO}_1$ for the real object of subsystem 1). Remember that something can be both an image and an object.
**c)** A ray $(y_1, \theta_1)$ passes through the optical system, beginning in the object plane. Find the ray $(y_2, \theta_2)$ in the image plane.

**Solution.**
**a)** The final image location is $\boxed{z_2' = 5f/3}$ and the total magnification is $\boxed{m = 2/3}$.

| Subsystem | Focal length | Object location | Image location | Magnification |
|---|---|---|---|---|
| Converging lens | $f_1 = f$ | $s_1 = f/2$ | $s_1' = -f$ | $m_1 = 2$ |
| Diverging lens | $f_2 = -2f$ | $s_2 = 4f$ | $s_2' = -4f/3$ | $m_2 = 1/3$ |

**b)** The system is sketched below. The three principal rays of both lenses are indicated with arrows.

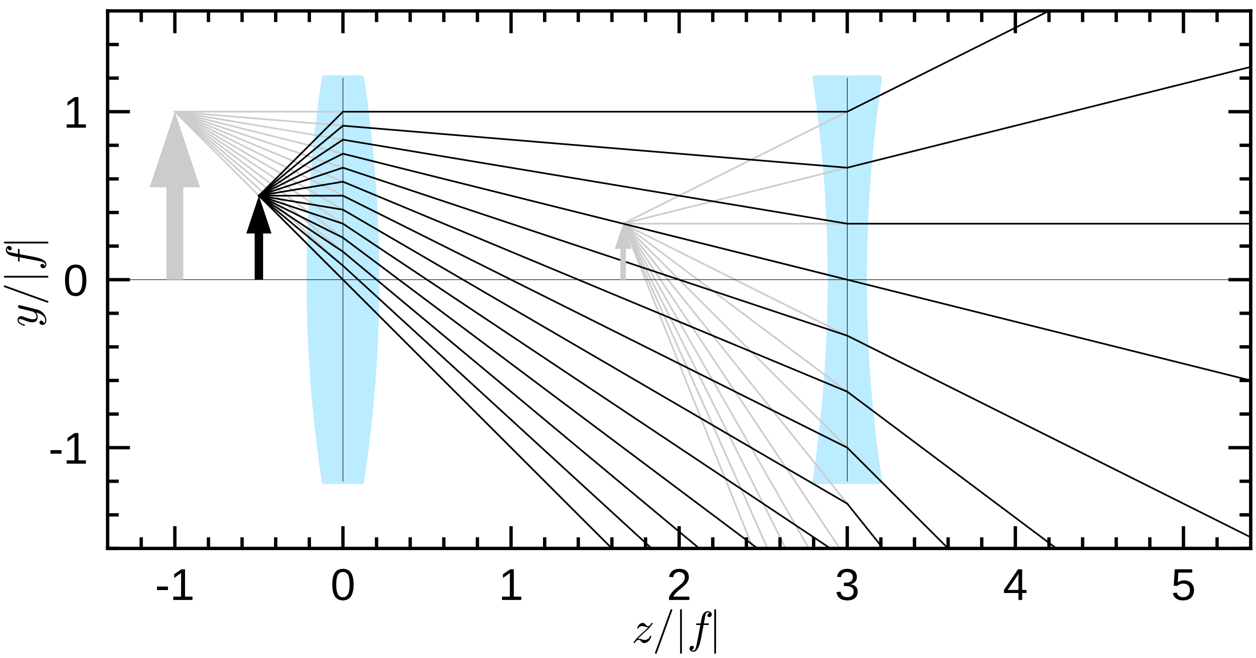



**c)** The system's net optical ray transfer matrix that traces rays from the mirror's object plane to the lens's image plane is

$$
\begin{aligned}
M &= M_2 \cdot M_1 \\
&= \begin{pmatrix} m_2 & 0 \\ -1/f_2 & 1/m_2 \end{pmatrix} \begin{pmatrix} m_1 & 0 \\ -1/f_1 & 1/m_1 \end{pmatrix} \\
&= \begin{pmatrix} 1/3 & 0 \\ 1/(2f) & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1/f & 1/2 \end{pmatrix} \\
&= \begin{pmatrix} 2/3 & 0 \\ -2/f & 3/2 \end{pmatrix}
\end{aligned}
$$

Then,

$$
\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = M \cdot \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} 2/3 & 0 \\ -2/f & 3/2 \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} = \boxed{\begin{pmatrix} 2y_1/3 \\ 3\theta_1/2 - 2y_1/f \end{pmatrix}}
$$




Shown above is a fan of rays traced through the system. The rays were traced using ray transfer matrices. Unlike the hand-tracing method, where we only trace principal rays through a single element, these rays have arbitrary angles and can be traced through the entire system.