Physics 110B - Midterm 2

Monday, March 7, 10-11AM, in PAB 1434A

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Solutions

1) The incident wave is

$$\mathbf{E}_I = \hat{\mathbf{x}} E_0 \cos(kz - \omega t) \quad \text{and} \quad \mathbf{B}_I = -\hat{\mathbf{y}}(E_0/c)\cos(kz - \omega t)$$

In the perfect conductor, there is no electromagnetic field. Because the tangential component of the electric field is continuous across the boundary, the tangential component of the electric field on the vacuum side of the boundary must be zero. To satisfy this, the reflected wave is

$$\mathbf{E}_R = -\hat{\mathbf{x}} E_0 \cos(-kz - \omega t) \quad \text{and} \quad \mathbf{B}_R = -\hat{\mathbf{y}}(E_0/c)\cos(-kz - \omega t)$$

At the boundary, the discontinuity in the tangential components of the magnetic field determines the induced surface current. Using the given identity, taking medium 1 to be the conductor and medium 2 to be the vacuum,

$$\hat{\mathbf{n}} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mu_0 \mathbf{K} \implies \hat{\mathbf{z}} \times [-\hat{\mathbf{y}}(2E_0/c)\cos\omega t] = \mu_0 \mathbf{K}$$

Then,

$$\boxed{\mathbf{K} = \hat{\mathbf{x}} \frac{2E_0}{\mu_0 c} \cos\omega t}$$

2) Following the recipe for TE modes, we obtain the longitudinal field for the TE_{10} mode,

$$B_z(x, y) = B_0 \cos\frac{\pi x}{a},$$

and k is determined through the dispersion relation

$$\omega^2 = c^2 k^2 + \frac{c^2 \pi^2}{a^2} \implies \boxed{k = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}}}$$

Note that we took the positive value of k, as appropriate for propagation in the positive z direction.

The transverse components of the electromagnetic field are determined from $B_z(x, y)$, using $\gamma_{10}^2 = (\pi/a)^2$, as

$$\mathbf{B}_t = \frac{ik}{\gamma_{10}^2} \nabla_t B_z = i\hat{\mathbf{x}} \frac{ka^2}{\pi^2} \frac{\partial B_z}{\partial x} = -i\hat{\mathbf{x}} \frac{B_0 k a}{\pi} \sin\frac{\pi x}{a}$$

and

$$\mathbf{E}_t = -\frac{\omega}{k} \hat{\mathbf{z}} \times \mathbf{B}_t = i\frac{\omega}{k} \frac{B_0 k a}{\pi} (\hat{\mathbf{z}} \times \hat{\mathbf{x}}) \sin\frac{\pi x}{a} = i\hat{\mathbf{y}} \frac{B_0 \omega a}{\pi} \sin\frac{\pi x}{a}$$

The full expression for the electromagnetic field within the waveguide is then

$$\mathbf{E} = -\hat{\mathbf{y}}\frac{B_0\omega a}{\pi}\sin\left(\frac{\pi x}{a}\right)\sin(kz-\omega t)$$
$$\mathbf{B} = \hat{\mathbf{z}}B_0\cos\left(\frac{\pi x}{a}\right)\cos(kz-\omega t) + \hat{\mathbf{x}}\frac{B_0 k a}{\pi}\sin\left(\frac{\pi x}{a}\right)\sin(kz-\omega t)$$

Note that the transverse components are 90 degrees out-of-phase from the longitudinal component.

The surface current density on the $x = 0$ wall is found from the given boundary condition identity, taking medium 1 to be the conductor and medium 2 to be the vacuum and $\hat{\mathbf{n}} = \hat{\mathbf{x}}$,

$$\hat{\mathbf{n}} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mu_0\mathbf{K} \implies \mathbf{K} = \frac{1}{\mu_0}(\hat{\mathbf{x}} \times \mathbf{B}) = \boxed{-\hat{\mathbf{y}}\frac{B_0}{\mu_0}\cos(kz-\omega t)}$$

Similarly, the surface charge density on the $y = 0$ wall is found from the given boundary condition identity, taking medium 1 to be the conductor and medium 2 to be the vacuum and $\hat{\mathbf{n}} = \hat{\mathbf{y}}$,

$$\hat{\mathbf{n}} \cdot (\mathbf{E}_2 - \mathbf{E}_1) = \frac{\sigma}{\epsilon_0} \implies \sigma = \epsilon_0\hat{\mathbf{y}} \cdot \mathbf{E} = \boxed{-\frac{B_0\epsilon_0\omega a}{\pi}\sin\left(\frac{\pi x}{a}\right)\sin(kz-\omega t)}$$

In these results, B_0 is an arbitrary overall amplitude.

3) The retarded scalar potential is

$$\begin{aligned}\phi(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}dV' \\ &= \frac{1}{4\pi\epsilon_0}\int_V\frac{\alpha(t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}dV' \\ &= \frac{\alpha t}{4\pi\epsilon_0}\int_V\frac{dV'}{|\mathbf{r}-\mathbf{r}'|} - \frac{\alpha}{4\pi\epsilon_0 c}\int_V dV'\end{aligned}$$

The V subscripts indicate that the integrals are to be taken over the sphere of radius a. The first term is simply the electrostatic potential of a sphere having uniform charge density αt, while the second integral is the volume of the sphere.

The most efficient way to calculate the electrostatic potential of a uniformly charged sphere is to radially integrate the electric field obtained from Gauss's law,

$$E_r(r) = \begin{cases}\dfrac{\alpha t r}{3\epsilon_0} & r < a \\ \dfrac{\alpha t a^3}{3\epsilon_0 r^2} & r > a\end{cases}$$

Then, outside the sphere,

$$\phi(\infty) - \phi(r) = \int_r^\infty \frac{\partial\phi}{\partial r}\,dr = -\int_r^\infty E_r(r)\,dr = -\frac{\alpha t a^3}{3\epsilon_0 r} \implies \phi(r) = \frac{\alpha t a^3}{3\epsilon_0 r},$$

and, inside the sphere,

$$\phi(a) - \phi(r) = -\int_r^a E_r(r)\,dr = -\frac{\alpha t}{6\epsilon_0}(a^2 - r^2) \implies \phi(r) = \frac{\alpha t}{6\epsilon_0}(3a^2 - r^2)$$

Finally, the retarded scalar potential of the given time-dependent charge distribution is

$$\boxed{\phi(r,t) = \begin{cases} \dfrac{\alpha t}{6\epsilon_0}(3a^2 - r^2) - \dfrac{\alpha a^3}{3\epsilon_0 c} & r < a \\ \dfrac{\alpha t a^3}{3\epsilon_0 r} - \dfrac{\alpha a^3}{3\epsilon_0 c} = \dfrac{a^3}{3\epsilon_0 r}\alpha(t - r/c) & r > a \end{cases}}$$

4) (S. Ebert) News of the current switching on, and its subsequent charge accumulation, doesn't arrive at the observation point until $t = x/c$. So, for $t < x/c$, $\phi(x,t) = 0$ and $\mathbf{A}(x,t) = 0$.

In the remaining, we assume $t > x/c$. Using

$$\rho(\mathbf{r},t) = q(t)\delta^3(\mathbf{r}) = \alpha t^2 \delta^3(\mathbf{r}),$$

the retarded scalar potential, evaluated at $\mathbf{r} = x\,\hat{\mathbf{x}}$, is

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}dV' = \boxed{\frac{1}{4\pi\epsilon_0}\frac{\alpha(t - x/c)^2}{x}}$$

The retarded vector potential, evaluated at $\mathbf{r} = x\,\hat{\mathbf{x}}$, is

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}dV' = \hat{\mathbf{x}}\,\frac{\mu_0}{4\pi}\int_{-\infty}^{0} \frac{I[t_r(x')]\,dx'}{x - x'}$$

The current at the retarded time $t_r = t - (x - x')/c$ is

$$I[t_r(x')] = \begin{cases} 2\alpha t_r & t_r > 0 \\ 0 & t_r < 0 \end{cases} = \begin{cases} 2\alpha[t - (x - x')/c] & x' > x - ct \\ 0 & x' < x - ct \end{cases}$$

Plugging in and then substituting $u = x - x'$,

$$\begin{aligned} \mathbf{A}(\mathbf{r},t) &= \hat{\mathbf{x}}\,\frac{\mu_0}{4\pi}\int_{x-ct}^{0} \frac{2\alpha\,[t - (x - x')/c]}{x - x'}\,dx' \\ &= -\hat{\mathbf{x}}\,\frac{\mu_0\alpha}{2\pi}\int_{ct}^{x} \frac{t - u/c}{u}\,du \\ &= -\hat{\mathbf{x}}\,\frac{\mu_0\alpha}{2\pi}\left[t\int_{ct}^{x}\frac{du}{u} - \frac{1}{c}\int_{ct}^{x} du\right] \\ &= \boxed{\hat{\mathbf{x}}\,\frac{\mu_0\alpha}{2\pi}\left[t\ln\left(\frac{ct}{x}\right) - (t - x/c)\right]} \end{aligned}$$