Physics 110B - Midterm 1

Friday, February 11, 10-11AM, in PAB 1434A

UCLA / Winter 2022 / Brian Naranjo

Solutions

1) (10 points) Taking the wire to lie along the z axis, the magnetic field is obtained from Ampère's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \implies B_\phi 2\pi s = \mu_0 I_0 \implies \mathbf{B} = \frac{\mu_0 I_0}{2\pi s}\hat{\boldsymbol{\phi}}$$

Taking the orientation of the square loop to be counterclockwise as shown, the magnetic flux through the square loop is

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \frac{\mu_0 I_0 a}{2\pi} \int_x^{x+a} \frac{ds}{s} = \frac{\mu_0 I_0 a}{2\pi} \ln\left(\frac{x+a}{x}\right)$$

As we pull the square loop away from the wire, the flux through the loop *decreases*, inducing an emf. Faraday's flux rule and the chain rule give

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d\Phi}{dx}\frac{dx}{dt} = \frac{\mu_0 I_0 v}{2\pi} \frac{a^2}{x(x+a)} = IR$$

Assuming $I_0 > 0$ and $v > 0$, we have $I > 0$ so that the induced current is counterclockwise. This agrees with Lenz's law in that a counterclockwise current acts to *increase* the flux through the square loop.

For a given steady current I, we should pull the square loop away with a velocity

$$v = \boxed{\frac{2\pi}{\mu_0}\frac{IR}{I_0}\frac{x(x+a)}{a^2}}$$

Note that this result is still true for a constant current flowing clockwise ($I < 0$). In this case, $v < 0$, and we are pushing the loop *towards* the wire.

2) (10 points) We are asked to find the induced current $I_2(t)$ in the rectangular loop due to a given current $I_1(t)$ flowing in the square loop. Faraday's flux rule gives

$$\mathcal{E}_{1\to 2} = -\frac{d\Phi_{1\to 2}}{dt} = -M_{21}\frac{dI_1}{dt} = I_2 R_2$$

Therefore, we just need to find the coefficient of mutual inductance M_{21}. Let's back up a step and write down how the coefficients of mutual inductance are defined,

$$\Phi_{1\to 2} = M_{21} I_1$$
$$\Phi_{2\to 1} = M_{12} I_2$$

To find M_{21}, we suppose that a current I_1 flows in the square loop and then calculate the resulting flux $\Phi_{1\to 2}$ through the rectangular flux. This is easier said than done. Not only is the square loop's magnetic field complicated, but, as there is no exploitable symmetry, finding the flux through the rectangular loop seems hopeless.

On the other hand, using the given assumption $a \ll b$, the other coefficient of mutual inductance, M_{12}, is trivial to calculate. Supposing a current I_2 flows in the rectangular loop, the flux through the square loop is

$$\Phi_{2\to 1} = 2\frac{\mu_0 I_2 a}{2\pi}\int_a^{2a}\frac{ds}{s} = \frac{\mu_0 I_2 a}{\pi}\ln 2 = M_{12}I_2,$$

which is essentially the same calculation as in Problem 1, and the factor of 2 is due to contributions from both of the long sides of the rectangular loop.

Then, using the reciprocity theorem of mutual inductance,

$$M_{21} = M_{12} = \frac{\mu_0 a\ln 2}{\pi} \equiv M$$

Finally,

$$I_2(t) = -\frac{M}{R_2}\frac{dI_1}{dt} = \boxed{-\frac{\mu_0 a\ln 2}{\pi R_2}\frac{dI_1}{dt}}$$

As a check, for $dI_1/dt > 0$, the flux through the rectangular loop is increasing. Therefore, the induced current through the rectangular loop should be in the clockwise direction to oppose this change in flux. This is consistent with the negative sign in the answer.

3) (10 points) To evaluate the Poynting vector everywhere, we will first need to find the electromagnetic fields everywhere.

The electric field between the cylinders is found from Gauss's law,

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \implies E_s 2\pi s L = \frac{\lambda L}{\epsilon_0} \implies \mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}.$$

Elsewhere, the electric field is zero.

The magnetic field between the cylinders is found from Ampère's law,

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \implies B_\phi 2\pi s = \mu_0 I \implies \mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}},$$

where $I = \lambda v$. Elsewhere, the magnetic field is zero.

Then, the Poynting vector is

$$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \boxed{\begin{cases}\dfrac{v\lambda^2}{4\pi^2\epsilon_0 s^2}\hat{\mathbf{z}} & \text{between the cylinders}\\ 0 & \text{elsewhere}\end{cases}}$$

The net rate of electromagnetic energy flowing through the xy plane is

$$P = \int_{z=0}(\mathbf{S}\cdot\hat{\mathbf{z}})\,dx\,dy = \frac{v\lambda^2}{4\pi^2\epsilon_0}2\pi\int_a^b\frac{ds}{s} = \boxed{v\frac{\lambda^2}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)}$$

4) (10 points) The net force acting on the charges and currents within a volume V bound by a surface S is

$$\mathbf{F}_{\text{mech}} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \int_V \frac{\partial \mathbf{g}}{\partial t} dV.$$

In this problem, everything is static, so we can drop the volume integral.

We need to **explicitly** specify an imaginary closed surface S that completely encloses point charge q_1, but **not** point charge q_2. Though any such choice will, in theory, produce the correct result, one choice that gives an easy calculation is to choose a hemispherical 'bowl,' of radius $R > a$, centered on the origin, together with its 'disk' base, lying in the xy plane.

We must first argue that, for $R \to \infty$,

$$\int_{\text{bowl}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} = 0.$$

It is sufficient to argue that, in the limit $R \to \infty$, the electric field scales as $1/R^2$, as the leading term in the multipole expansion is the monopole term. Then, the bowl surface's area scales as R^2 while the Maxwell stress tensor scales as $1/R^4$ – giving an overall scaling of $1/R^2$ for the value of the integral.

We need to calculate the Maxwell stress tensor in the xy plane. Taking $\mathbf{r} = s\hat{\mathbf{s}}$, the relative position vectors are

$$\mathbf{R}_1 = \mathbf{r} - \mathbf{r}_1 = s\hat{\mathbf{s}} - a\hat{\mathbf{z}}$$
$$\mathbf{R}_2 = \mathbf{r} - \mathbf{r}_2 = s\hat{\mathbf{s}} + a\hat{\mathbf{z}}$$

The electric field in the xy plane is then

$$\mathbf{E} = \frac{q_1 \mathbf{R}_1}{4\pi\epsilon_0 R_1^3} + \frac{q_2 \mathbf{R}_2}{4\pi\epsilon_0 R_2^3} = \frac{q}{2\pi\epsilon_0} \frac{s\hat{\mathbf{s}}}{(s^2 + a^2)^{3/2}}$$

In the xy plane, $E_z = 0$, so that $T_{xz} = T_{yz} = 0$, and $\overleftrightarrow{\mathbf{T}} \cdot \hat{\mathbf{z}} = T_{zz}\hat{\mathbf{z}}$, where

$$T_{zz} = \epsilon_0 \left[E_z E_z - \frac{1}{2} E^2 \right] = -\frac{\epsilon_0}{2} E^2 = -\frac{q^2}{8\pi^2\epsilon_0} \frac{s^2}{(s^2 + a^2)^3}$$

Then, we have, taking $d\mathbf{a} = -dx\, dy\, \hat{\mathbf{z}} = -2\pi s\, ds\, \hat{\mathbf{z}}$ to be the *outward* surface normal at the disk,

$$\mathbf{F}_{\text{mech}} = \int_{\text{disk}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} = -\hat{\mathbf{z}}\, 2\pi \int_0^\infty T_{zz}\, s\, ds = \hat{\mathbf{z}} \frac{q^2}{4\pi\epsilon_0} \int_0^\infty \frac{s^3\, ds}{(s^2 + a^2)^3}$$

The following definite integral was provided on the blackboard,

$$\int_0^\infty \frac{s^3\, ds}{(s^2 + a^2)^3} = \frac{1}{4a^2}$$

Then,

$$\mathbf{F}_{\text{mech}} = \boxed{\frac{q^2}{4\pi\epsilon_0 (2a)^2} \hat{\mathbf{z}}}$$