Physics 110B - Final

Thursday, March 17, 8-11AM, in PAB 1434A

UCLA / Winter 2022 / Brian Naranjo

Solutions

1) Using Ampère's law, the solenoid's magnetic field is

$$\mathbf{B}(s,t) = \begin{cases} \mu_0 n I(t)\,\hat{\mathbf{z}} & s < a \\ 0 & s > a \end{cases}$$

The magnetic flux through a disc of radius s, lying in the xy plane, centered on the origin, and oriented in the positive z direction, is

$$\Phi(s,t) = \begin{cases} (\pi s^2)\mu_0 n I_0 \cos\omega t & s < a \\ (\pi a^2)\mu_0 n I_0 \cos\omega t & s \geq a \end{cases}$$

Then, Faraday's law gives the electric field in terms of the changing flux,

$$2\pi s E_\phi = -\frac{\partial \Phi}{\partial t} \implies \mathbf{E} = \boxed{\begin{cases} \hat{\boldsymbol{\phi}}\,\dfrac{s}{2}(\mu_0 n I_0 \omega)\sin\omega t & s < a \\ \hat{\boldsymbol{\phi}}\,\dfrac{a^2}{2s}(\mu_0 n I_0 \omega)\sin\omega t & s \geq a \end{cases}}$$

2) The electric field is radially directed outward from the origin,

$$\mathbf{E}(t) = \frac{Q(t)}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

Then, the displacement current through the sphere is

$$I_d = \oint \mathbf{J}_d \cdot d\mathbf{a} = \epsilon_0 \oint \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} = \frac{dQ}{dt} = \boxed{I(t)}$$

3) The net force acting on the charges and currents within a volume V bound by a surface S is

$$\mathbf{F}_{\text{mech}} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \int_V \frac{\partial \mathbf{g}}{\partial t} dV.$$

In this problem, everything is static, so we can drop the volume integral.

We need to **explicitly** specify an imaginary closed surface S that completely encloses the upper shell, but **not** the lower shell. Though any such choice will, in theory, produce the correct result, one choice that gives an easy calculation is to choose a hemispherical 'bowl,' of radius $R > a$, centered on the origin, together with its 'disk' base, lying in the xy plane.

We must first argue that, for $R \to \infty$,

$$\int_{\text{bowl}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} = 0.$$

It is sufficient to argue that, in the limit $R \to \infty$, the electric field scales as $1/R^2$, as the leading term in the multipole expansion is the monopole term. Then, the bowl surface's area scales as R^2 while the Maxwell stress tensor scales as $1/R^4$ – giving an overall scaling of $1/R^2$ for the value of the integral.

Then, it remains to calculate the *downward* flux through the xy plane. Inside the sphere, the electric field is zero, so that the Maxwell stress tensor is zero. Outside the sphere, the electric field is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\hat{\mathbf{r}}.$$

Noting that, in the xy plane, $E_z = 0$, which then implies $T_{xz} = T_{yz} = 0$ and $T_{zz} = -(1/2)\epsilon_0 E^2$. So, the electrostatic force acting on the upper shell is

$$F_{\text{upper}} = \hat{\mathbf{z}} \cdot \int_{xy} [\overleftrightarrow{\mathbf{T}} \cdot (-\hat{\mathbf{z}})]\, da = -\int_{xy} T_{zz}\, da = \frac{Q^2}{16\pi\epsilon_0}\int_a^\infty \frac{dR}{R^3} = \frac{Q^2}{32\pi\epsilon_0 a^2}$$

The force required to keep the upper shell in place is equal and opposite,

$$\boxed{\mathbf{F}_{\text{push}} = -\frac{Q^2}{32\pi\epsilon_0 a^2}\hat{\mathbf{z}} = -\frac{(Q/2)^2}{4\pi\epsilon_0 (a\sqrt{2})^2}\hat{\mathbf{z}}}$$

The final expression shows that the magnitude of the force is equal to the force between two point charges $Q/2$ separated by a distance $a\sqrt{2}$.

4) The current flow is steady, so the continuity equation gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \implies 0 = \nabla \cdot (\sigma \mathbf{E}) = \frac{1}{s}\frac{\partial}{\partial s}[s\sigma(s)E_s] \implies \frac{\partial^2 V}{\partial s^2} = 0.$$

Therefore, the potential $V(s)$ varies linearly between the inner and outer cylindrical shells,

$$V(s) = V_0\left(\frac{b-s}{b-a}\right)$$

Going back to the continuity equation,

$$0 = \sigma(\nabla \cdot \mathbf{E}) + \mathbf{E} \cdot \nabla\sigma \implies \rho = -\frac{\epsilon_0}{\sigma}\mathbf{E} \cdot \nabla\sigma = \frac{\epsilon_0}{\sigma}\left(\frac{\partial V}{\partial s}\right)\left(\frac{\partial \sigma}{\partial s}\right) = \boxed{\frac{\epsilon_0}{s}\left(\frac{V_0}{b-a}\right)}$$

5) By the same reasoning in deriving the Fresnel coefficients, the reflected wavevector is in the plane of incidence and the angle of reflection equals the angle of incidence. Then,

$$\mathbf{k}_R = k\hat{\mathbf{k}}_R = k(\hat{\mathbf{y}}\sin\theta + \hat{\mathbf{z}}\cos\theta)$$

At the interface, the tangential component of the electric field must vanish, therefore, the reflected electric field is

$$\mathbf{E}_R = -\hat{\mathbf{x}} E_0 \cos(\mathbf{k}_R \cdot \mathbf{r} - \omega t)$$

The total magnetic field at $z = 0$ is

$$\begin{aligned}
\mathbf{B} = \mathbf{B}_I + \mathbf{B}_R &= \frac{1}{c}\hat{\mathbf{k}}_I \times \mathbf{E}_I + \frac{1}{c}\hat{\mathbf{k}}_R \times \mathbf{E}_R = \frac{1}{c}(\hat{\mathbf{k}}_I - \hat{\mathbf{k}}_R) \times \mathbf{E}_I \\
&= \left[-\frac{2\cos\theta}{c}\hat{\mathbf{z}}\right] \times [\hat{\mathbf{x}} E_0 \cos(ky\sin\theta - \omega t)] \\
&= -\hat{\mathbf{y}}\frac{2E_0\cos\theta}{c}\cos(ky\sin\theta - \omega t)
\end{aligned}$$

The surface current on the interface is then

$$\mathbf{K} = \frac{1}{\mu_0}\hat{\mathbf{z}} \times \mathbf{B} = \boxed{\hat{\mathbf{x}}\frac{2E_0\cos\theta}{\mu_0 c}\cos(ky\sin\theta - \omega t)}$$

For $\theta = 0$, this reduces to Problem 1 of MT2.

6) We take take the transverse electric field to be the same form as the field between infinite parallel plates,

$$\mathbf{E}_{\text{TEM}} = E_0\,\hat{\mathbf{x}}.$$

We confirm that it satisfies both the electrostatic requirements

$$\nabla_t \cdot \mathbf{E}_{\text{TEM}} = 0 \quad \text{and} \quad \nabla_t \times \mathbf{E}_{\text{TEM}} = 0$$

while also satisfying $\mathbf{E}^{\|}_{\text{TEM}} = 0$ at the boundaries.

From our TEM recipe (L15), the transverse magnetic field is

$$\mathbf{B}_{\text{TEM}} = \frac{1}{c}\hat{\mathbf{z}} \times \mathbf{E}_{\text{TEM}} = \frac{E_0}{c}\hat{\mathbf{y}}$$

The full expressions for the electric and magnetic fields within the waveguide are then

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_0 \cos(kz - \omega t) \quad \text{and} \quad \mathbf{B}(z,t) = \hat{\mathbf{y}}(E_0/c)\cos(kz - \omega t).$$

From the dispersion relation, $\omega^2 = c^2k^2$, we take positive $\boxed{k = \omega/c}$, as appropriate for propagation in the positive z direction.

At the wall $x = a$, the surface normal pointing from the metal into the vacuum is $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$. Then, the surface charge density is

$$\sigma(z,t) = \epsilon_0 \hat{\mathbf{n}} \cdot \mathbf{E}(z,t) = \boxed{-\epsilon_0 E_0 \cos(kz - \omega t)}$$

and the surface current density is

$$\mathbf{K}(z,t) = \frac{1}{\mu_0}\hat{\mathbf{n}} \times \mathbf{B}(z,t) = \boxed{-\hat{\mathbf{z}}\frac{E_0}{\mu_0 c}\cos(kz - \omega t)}$$

7) Case $t < a/c$:
The observer at the origin hasn't received news of the flowing current. Therefore, all the fields are zero.

Case $a/c < t < b/c$:
For a filamentary current, the retarded vector potential is

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}\,dV' = \frac{\mu_0}{4\pi}\int \frac{I[t_r(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|}\,d\mathbf{s}'$$

In terms of the source location, the current is

$$I[t_r(\mathbf{r}')] = \begin{cases} I_0 & t > 0 \\ 0 & t < 0 \end{cases} = \begin{cases} I_0 & |\mathbf{r}'| < ct \\ 0 & \text{elsewhere} \end{cases}$$

Within this time range, the small semicircle always gives its full contribution,

$$\mathbf{A}_a(t) = \frac{\mu_0}{4\pi}\frac{I_0}{a}\int_a d\mathbf{s}' = \frac{\mu_0 I_0}{4\pi a}(2a\hat{\mathbf{x}}) = \frac{\mu_0 I_0}{2\pi}\hat{\mathbf{x}}.$$

The two straight segments, each giving equal contributions, yield a total

$$\mathbf{A}_{ab}(t) = 2\frac{\mu_0}{4\pi}I_0\hat{\mathbf{x}}\int_a^{ct}\frac{dx'}{x'} = \frac{\mu_0 I_0}{2\pi}\ln\left(\frac{ct}{a}\right)\hat{\mathbf{x}}$$

The large semicircle doesn't contribute in this time range. Therefore, the vector potential at the origin is

$$\boxed{\mathbf{A}(t) = \frac{\mu_0 I_0}{2\pi}\left[1+\ln\left(\frac{ct}{a}\right)\right]\hat{\mathbf{x}}}$$

Because the wire remains neutral, the scalar potential is always zero, and the electric field at the origin is

$$\boxed{\mathbf{E}(t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 I_0}{2\pi t}\hat{\mathbf{x}}}$$

Case $b/c < t$:
In this time range, the entire loop gives its full contribution.

The large semicircle's contribution,

$$\mathbf{A}_b(t) = \frac{\mu_0}{4\pi}\frac{I_0}{b}\int_b d\mathbf{s}' = \frac{\mu_0 I_0}{4\pi b}(-2b\hat{\mathbf{x}}) = -\frac{\mu_0 I_0}{2\pi}\hat{\mathbf{x}},$$

cancels the small semicircle's contribution.

Then, the vector potential at the origin is

$$\boxed{\mathbf{A}(t) = \frac{\mu_0 I_0}{2\pi}\ln\left(\frac{b}{a}\right)\hat{\mathbf{x}}}$$

The vector potential is now time-independent, so

$$\boxed{\mathbf{E}(t) = -\frac{\partial \mathbf{A}}{\partial t} = 0}$$

Case $t = a/c$:
At this time, the vector potential abruptly jumps from zero to

$$\mathbf{A}_0 = \frac{\mu_0 I_0}{2\pi}\hat{\mathbf{x}}.$$

We can express this mathematically using the unit step function,

$$\mathbf{A}(t) = \mathbf{A}_0\ \theta(t - a/c)$$

Then, in the neighborhood of $t = a/c$, using $d\theta/dx = \delta(x)$,

$$\mathbf{E}(t) = -\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{A}_0\delta(t - a/c) = \boxed{-\hat{\mathbf{x}}\frac{\mu_0 I_0}{2\pi}\delta(t - a/c)}$$

Realistically, the loop has a self-inductance so that the current can not be discontinuous. Furthermore, the wire will have a finite cross-section. Therefore, in reality, at the origin, we would briefly see a large, but finite, electric field. However, we *can* reasonably say what impulse a charge q at the origin at time $t = a/c$ would receive,

$$\Delta\mathbf{p} = \int \mathbf{F}\,dt = \int q\mathbf{E}\,dt = -\hat{\mathbf{x}}\frac{\mu_0 I_0 q}{2\pi}$$

Case $t = b/c$:
At this time, the vector potential abruptly jumps back down, undoing the previous jump. Therefore, in the neighborhood of $t = b/c$,

$$\mathbf{E}(t) = -\frac{\partial \mathbf{A}}{\partial t} = +\mathbf{A}_0\delta(t - b/c) = \boxed{+\hat{\mathbf{x}}\frac{\mu_0 I_0}{2\pi}\delta(t - b/c)}$$

8) At time $t = 0$, by symmetry, the dipole moment is along the z-axis. The magnitude of the dipole moment is

$$p_0 = \hat{\mathbf{z}}\cdot\mathbf{p}(t=0) = \rho\int_{\text{hemi}} z\,dV = 2\pi\rho\int_0^b r^2\,dr\int_0^1 d\xi\ (r\xi) = \rho\frac{\pi b^4}{4}$$

In terms of the hemisphere's charge $Q = \rho(2\pi b^3/3)$, the dipole moment is

$$\boxed{p_0 = \frac{3}{8}Qb}$$

Then, the rotating dipole moment is

$$\mathbf{p}(t) = p_0(\hat{\mathbf{z}}\cos\omega t - \hat{\mathbf{y}}\sin\omega t), \quad \text{and} \quad \ddot{\mathbf{p}}(t) = -p_0\omega^2(\hat{\mathbf{z}}\cos\omega t - \hat{\mathbf{y}}\sin\omega t)$$

In both of the following cases, $\boxed{t_r = t - r/c}$.

At $\mathbf{r} = x\,\hat{\mathbf{x}}$,

$$\mathbf{E}(\mathbf{r},t) = \frac{\mu_0}{4\pi x}[\hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \ddot{\mathbf{p}}(t_r))] = \boxed{\frac{\mu_0 p_0 \omega^2}{4\pi x}(-\hat{\mathbf{y}}\sin\omega t_r + \hat{\mathbf{z}}\cos\omega t_r)}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\,\hat{\mathbf{x}} \times \mathbf{E} = \boxed{\frac{\mu_0 p_0 \omega^2}{4\pi x c}(-\hat{\mathbf{y}}\cos\omega t_r - \hat{\mathbf{z}}\sin\omega t_r)}$$

At $\mathbf{r} = y\,\hat{\mathbf{y}}$,

$$\mathbf{E}(\mathbf{r},t) = \frac{\mu_0}{4\pi y}[\hat{\mathbf{y}} \times (\hat{\mathbf{y}} \times \ddot{\mathbf{p}}(t_r))] = \boxed{\frac{\mu_0 p_0 \omega^2}{4\pi y}\hat{\mathbf{z}}\cos\omega t_r}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\,\hat{\mathbf{y}} \times \mathbf{E} = \boxed{\frac{\mu_0 p_0 \omega^2}{4\pi y c}\hat{\mathbf{x}}\cos\omega t_r}$$

Along the x-axis, the radiation is circularly polarized, and, along the y-axis, the radiation is linearly polarized.